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Intrinsic Fluctuations and a Phase Transition in a Class of Large Populations of Interacting Oscillators

Hiroaki Daido[1]

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A theory of intrinsic fluctuations is developed of a phase ordering parameter for large populations of weakly and uniformly coupled limit-cycle oscillators with distributed native frequencies. In particular it is shown that the intensity as well as the correlation time of fluctuations exhibit power-law divergence at the onset of mutual entrainment with critical exponents which depend on whether the coupling strength approaches the threshold from below or above. This peculiar feature is demonstrated by numerical simulations mainly through finite-size scaling analyses. In the course of exploring its origin, we encounter a new concept termed a "correlation frequency" which provides a natural interpretation of the finite-size scaling laws. A comment is given on a recent theory by Kuramoto and Nishikawa to clarify why it contradicts our results.

KEY WORDS: Limit cycle; large degrees of freedom; entrainment; fluctuations; phase transition; scaling.

1. INTRODUCTION

Periodic oscillation is one of the typical behaviors exhibited by far-from-equilibrium systems encountered in a variety of fields of science, such as chemistry, biology, and engineering, as well as physics.[1-3] In general, it is possible to regard such a system as a large assembly of limit-cycle oscillators interacting with one another. Namely, every element of the system itself is of oscillatory nature, with its own rhythm. Then it would be quite natural to ask what the mechanism is for these "microscopic" rhythms to cooperatively yield the "macroscopic" temporal coherence dis-

[1] Department of Physics, Faculty of Engineering, Kyushu Institute of Technology, Tobata, Kitakyushu 804, Japan.

played by the whole system. Since the native frequency is considered to vary from one element to another in any realistic system, the question is never trivial, but may provide an interesting challenge to promote a new development in far-from-equilibrium statistical mechanics as well as in dynamical systems theory. Examples of rhythmic behavior may be most easily found in biological or physiological contexts, e.g., flickering of swarms of fireflies, peristaltic motion of gastrointestinal tracts, and fish locomotion.(3) Therefore, by elucidating the mechanism, we may be able to arrive at a deep insight into the nature of those so-called "biological rhythms" that are quite ubiquitous.

Aiming ultimately at the same goal, quite a few workers have recently carried out investigations into the properties of large populations of dissipative oscillators either analytically or numerically.(3–24) Among other things, it has been found that the onset of a macroscopic rhythm or mutual entrainment which occurs when the strength of interaction exceeds a threshold is quite analogous to second-order phase transitions in equilibrium cooperative systems, as first pointed out by Winfree.(4) In fact, some "order parameters" were introduced and shown to behave like those in conventional phase transitions.(25) Clearly, it is impossible to answer the question mentioned earlier without a complete understanding of such a new type of critical phenomenon, to which the present work is devoted.

A number of models have been used in theoretical investigations, which include coupled Van der Pol oscillators(7,9) and arrays of TDGL-type equations.(6,22,24) Among them, what we call "phase models"(3–5,8,10–21) constitute the simplest but nontrivial category, whose form is most generally as follows:

$$d\theta_j/dt = \Omega_j + \sum_{i=1}^{N} h_{ij}(\theta_i - \theta_j) \qquad (j = 1, \ldots, N) \tag{1.1}$$

where θ_j is the phase of the jth oscillator (in units of 2π), Ω_j the native frequency, and $h_{ij}(\theta)$ a coupling function such that $h_{ij}(\theta + 1) = h_{ij}(\theta)$. Though highly simplified, these equations can be derived in quite a general way under the conditions of weak coupling among elements as well as small dispersion of native frequencies.(8,10) This fact may suggest the utility of phase models in diverse areas. For instance, they were used to investigate peristaltic motion of intestines(11,12) and fish locomotion.(8)

Thanks to their apparent simplicity, phase models are particularly useful for studies with emphasis on the aspect of critical phenomena. For example, recently, use has been made of phase models with finite-range interactions to clarify especially the lattice dimensionality dependence of the onset of macroscopic entrainment.(18–21) In this paper we are concerned with another important class of phase models in which coupling among elements is of an infinite range:

$$d\theta_j/dt = \Omega_j + (\varepsilon/2\pi N) \sum_{i=1}^{N} \sin 2\pi(\theta_i - \theta_j) \tag{1.2}$$

(hereafter we omit the range of j as long as it does not differ from $1 \leqslant j \leqslant N$), where ε is the control parameter and the native frequencies are assumed to be distributed over the whole population, whose density we denote by $f(\Omega)$. The model, which was first studied by Kuramoto more than a decade ago,(5) may occupy a special place in the theory of population dynamics of oscillators, since it allows us to perform analytic investigations to some extent, thus facilitating our understanding of mutual entrainment on the firm basis of some exact results. At first sight the all-to-all interaction may seem to lack reality. It is known, however, that in multicellular organisms, for example, cell-to-cell coupling is not rarely long ranged (especially when it is due to electrical interactions via a "gap junction" of a high coupling ratio).(26) Probably there would be many other examples of a realistic system with far-reaching coupling among constituents. For any of them the model (1.2) should be useful at least as a first approximation.

Let us review what has been done so far for the model (1.2). A key quantity is the following phase-ordering parameter:

$$Z(t) = N^{-1} \sum_{j=1}^{N} e^{2\pi i'(\theta_j - \hat{\Omega} t)} \tag{1.3}$$

where $i' \equiv (-1)^{1/2}$ and $\hat{\Omega}$ is the frequency of entrainment. On the basis of a hypothesis that it has a limit for $t \to \infty$ as

$$\hat{Z} = \lim_{t \to \infty} Z(t) \tag{1.4}$$

in the *infinite-size* system, Kuramoto derived a self-consistent equation of the limit $\hat{Z}$, which enabled him to locate the threshold of mutual entrainment, hereafter denoted by ε_c, as well as to obtain its solutions.(5,10) The role of an order parameter is played by $\hat{Z}$, which obeys a scaling law as

$$|\hat{Z}| \propto (\varepsilon - \varepsilon_c)^{\beta} \tag{1.5}$$

immediately above the threshold, where the critical exponent β is 1/2 for typical $f(\Omega)$, which is not novel at all if the mean-field character of the model is taken into account.(25) (Recently Sakaguchi and Kuramoto extended the theory to the case of asymmetric couplings.(14)) It should be emphasized that the theory concerns the infinite-size system where $Z(t)$ exhibits no fluctuation. Once N is set to be finite, the Kuramoto theory is no longer applicable, because of undamped variation of $Z(t)$. In fact, a few years ago, Daido observed fairly strong sustained fluctuations of the macroscopic variable by numerical simulations using the model

$$\theta_j^{(n+1)} = \theta_j^{(n)} + \Omega_j + (\varepsilon/2\pi N) \sum_{i=1}^{N} \sin 2\pi(\theta_i^{(n)} - \theta_j^{(n)}) \tag{1.6}$$

for $N = 100$ and for a Lorentzian distribution of Ω_j with the half-value width of $\gamma = 10^{-3}$.[15] This model was originally proposed to investigate the dynamics in large populations of coupled quasiperiodic oscillators,[15] but it may be regarded as an Euler approximation for Eq. (1.2) for such a narrow distribution of the native frequency as chosen in ref. 15 so that we may interpret Daido's simulations as for the model (1.2). [The typical behavior of $Z(t)$ is displayed in Figs. 1 and 2. See also Fig. 1 of ref. 15.]

What was most remarkable in ref. 15 was the discovery of anomalous enhancement of fluctuations in the neighborhood of ε_c, which suggests the critical divergence of σ defined by

$$\sigma = \lim_{N \to \infty} (N \langle |Z - \langle Z \rangle|^2 \rangle)^{1/2} \tag{1.7}$$

where the pair of brackets stands for a long-time average. Subsequently Daido put forth evidence of the critical singularity of σ as

$$\sigma \propto |\varepsilon - \varepsilon_c|^{-\gamma'} \tag{1.8}$$

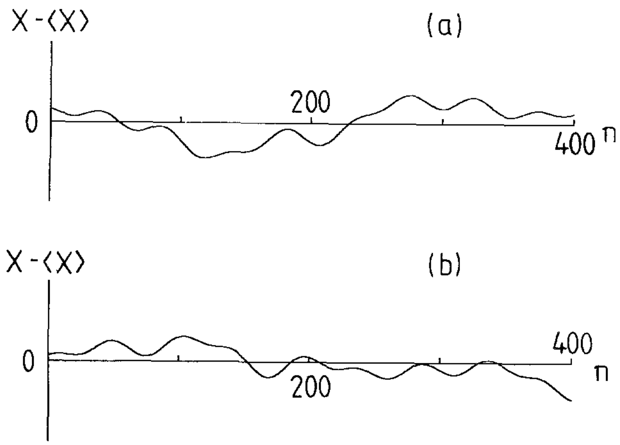

Fig. 1. Typical examples of persistent fluctuations of $X_n = \mathrm{Re}\, Z_n$ for $N = 100$ and for a Lorentzian distribution of Ω_j with $\gamma = 10^{-3}$ [see (3.11)]. The ordinates show $x \equiv X - \langle X \rangle$ for the range of $|x| < 0.25$. (a) $\varepsilon = 0.01$ ($< \varepsilon_c = 0.0125...$); (b) $\varepsilon = 0.015$ ($> \varepsilon_c$).

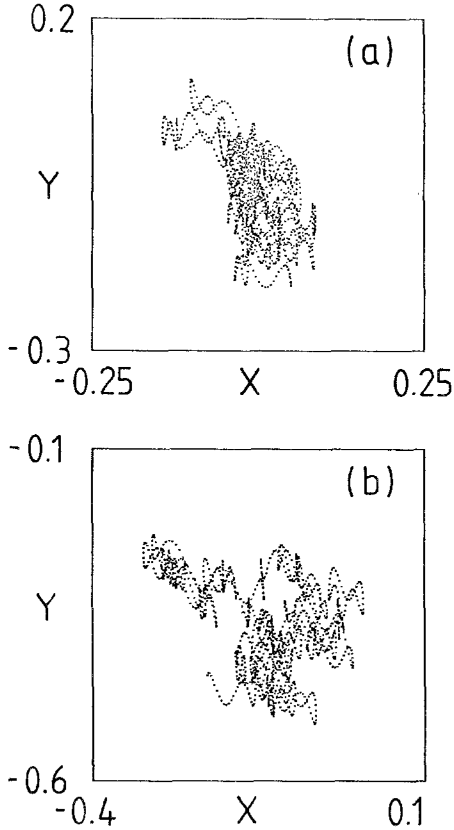

Fig. 2. Typical examples of the behavior of Z_n on the complex plane. In each box 1500 consecutive points are plotted. Other details are the same as in Fig. 1.

on both sides of the critical point by numerical computations for larger N (up to 1600).[16] Later, in contradiction with these observations, Kuramoto and Nishikawa claimed analytically that σ is not divergent at ε_c irrespective of the side from which the threshold is approached.[27]

Apart from which is true, let us consider here why the critical behavior of σ is important to begin with. For this purpose it may be instructive to compare our oscillator system with typical cooperative systems, say a magnet, whose static susceptibility corresponds to σ^2 in our system and exhibits critical divergence like (1.8) (with ε replaced by temperature), playing an important role in characterizing a phase transition.[25] Therefore how σ acts near ε_c may also be of importance to capture the nature of phase transitions in oscillator assemblies. This may be realized from the following fact as well: any novel feature of the phase transition could only be expected in the behavior of fluctuations since the order parameter exhibits stereotypical behavior alone, as we have already seen.

The purpose of this paper is twofold: (a) we develop a theory of fluctuations of the macroscopic variable Z for large populations as modeled by Eq. (1.2), hoping that it will make a meaningful step toward the statistical mechanics of oscillator populations; (b) then we clarify the critical behavior of fluctuations to search for unique features of the phase transition in comparison with conventional ones as well as to settle the controversy. The main results are as follows:

(i) σ and other related quantities indeed undergo divergence as (1.8) at the onset of mutual entrainment.

(ii) Their critical exponents, however, depend on whether the threshold is approached from below or above. This peculiar feature stems from the fact that elements of the system are not identical to one another, because of their native frequencies.

The paper is organized as follows: in the next section the basic framework of the theory is set up; then Sections 3 and 4 are devoted to detailed analyses for the subcritical and supercritical regimes, respectively; in Section 5 numerical evidence is given for the results after a discussion of finite-size scaling; in Section 6 the origin of the unique feature is pointed out and shown to lead to a new concept which is powerful in the interpretation of critical scalings; finally, in Section 7, the paper ends with a summary as well as a discussion and comments; some technical results are given in the appendices.[2]

2. BASIC FRAMEWORK

2.1. Behavior of the Order Parameter $\hat{Z}$

For convenience let us briefly review how $\hat{Z}$ can be evaluated analytically in the limit $N \to \infty$.[10] Throughout this paper we confine ourselves to the case that $f(\Omega)$ is symmetric with respect to a certain value, say $\tilde{\Omega}$. This implies that mutual entrainmernt takes place with the common frequency $\tilde{\Omega}$. With a set of new variables $\tilde{\theta}_j = \theta_j - \tilde{\Omega} t$, we see that Eq. (1.2) is converted into

$$d\tilde{\theta}_j/dt = \Delta_j + (\varepsilon/2\pi)\,\mathrm{Im}\{Z(t)e^{-2\pi i\tilde{\theta}_j}\} \tag{2.1}$$

[2] Some of the results in this paper were briefly reported in ref. 28.

where $\Delta_j \equiv \Omega_j - \tilde{\Omega}$ and Im stands for the imaginary part. For $N = \infty$, $Z(t)$ may be replaced by $\hat{Z}$ after an initial transient period [see (1.4)]. Then members of the population may be divided into two groups. One consists of oscillators with Δ_j such that

$$|\Delta_j| \leqslant (\varepsilon/2\pi)|\hat{Z}| \tag{2.2}$$

for which $\tilde{\theta}_j$ converges for $t \to \infty$ to

$$\alpha + \frac{1}{2\pi} \sin^{-1}\left(\frac{2\pi\Delta_j}{\varepsilon\,|\hat{Z}|}\right) \equiv \tilde{\theta}_j^* \tag{2.3}$$

where $\alpha \equiv (\operatorname{Arg} \hat{Z})/2\pi$. Namely, members of this group oscillate with the common frequency $\tilde{\Omega}$ and will be hereafter called "entrained oscillators." The other is formed by remaining members called "nonentrained oscillators," for which

$$|\Delta_j| > (\varepsilon/2\pi)|\hat{Z}| \tag{2.4}$$

and $\tilde{\theta}_j$ is periodic (mod 1) with a period given by

$$\begin{aligned} T_j &= \int_0^{\operatorname{sgn}(\Delta_j)} d\tilde{\theta}_j (d\tilde{\theta}_j/dt)^{-1} \\ &= \{\Delta_j^2 - (\varepsilon|\hat{Z}|/2\pi)^2\}^{-1/2} \end{aligned} \tag{2.5}$$

where $\operatorname{sgn}(x) \equiv x/|x|$.

Now let us consider the order parameter, to which the group of entrained oscillators alone contributes because of the assumed symmetry in $f(\Omega)$: by (1.3) and (1.4)

$$\hat{Z} = \int_{|\Delta| < (\varepsilon/2\pi)|\hat{Z}|} d\Delta\, \tilde{f}(\Delta) e^{2\pi i \tilde{\theta}^*}$$

where $\tilde{f}(\Delta) \equiv f(\Omega + \Delta)$. Using (2.3), we obtain the self-consistent equation of $\hat{Z}$ as follows:

$$|\hat{Z}| = 2\int_0^{(\varepsilon/2\pi)|\hat{Z}|} d\Delta\, \tilde{f}(\Delta) \left\{1 - \left(\frac{2\pi\Delta}{\varepsilon|\hat{Z}|}\right)^2\right\}^{1/2} \tag{2.6}$$

on the basis of which it is possible to show that a nontrivial solution bifurcates from a trivial one, $\hat{Z} = 0$, when ε exceeds

$$\varepsilon_c = 4/\tilde{f}(0) \tag{2.7}$$

beyond which a finite fraction of elements display a common periodicity by (2.2). Thus we have a phase transition at $\varepsilon = \varepsilon_c$. [*Remark.* The stability of the solutions to Eq. (2.6) remains to be proved, though much numerical evidence exists for it.[10,14–16] A first attempt in this direction is made in ref. 29.] Equation (2.6) reveals that the parameter has a power law growth near ε_c as

$$|\hat{Z}| \cong \frac{\pi \tilde{f}(0)^2}{[-2\tilde{f}''(0)]^{1/2}} (\varepsilon - \varepsilon_c)^{1/2} \tag{2.8}$$

where here and hereafter $f(\Omega)$ is assumed to be parabolic at $\Omega = \tilde{\Omega}$, as is the case with typical distributions. This is why we have $\beta = 1/2$ in (1.5). A careful check of this exponent was carried out in ref. 16.

2.2. Theory of Fluctuations

It is clear that the above theory is made successful by the absence of persistent fluctuations of Z in the limit $N \to \infty$. In what follows we aim at developing a theory to deal with the statistical behavior of the macroscopic quantity in large but finite-size systems.

To begin with, let us introduce a new variable w by

$$w(t) = Z(t) - \hat{Z} \tag{2.9}$$

which is small when N is large, provided we are concerned only with the regime following an initial transient period. (Suppose hereafter that $t = 0$ is included in that regime.) Then we may divide $\tilde{\theta}_j$ into two parts,

$$\tilde{\theta}_j = \psi_j + \phi_j \tag{2.10}$$

where ψ_j stands for dominant phase motion defined by

$$\begin{aligned} d\psi_j/dt &= \Delta_j + (\varepsilon/2\pi)\,\mathrm{Im}\{\hat{Z}e^{-2\pi i \psi_j}\} \\ \psi_j(0) &= \tilde{\theta}_j(0) = \theta_j(0) \end{aligned} \tag{2.11}$$

that is, ψ_j undergoes phase motion in the infinite-size system, and ϕ_j represents a deviation from it induced by w.

Now let us consider the order of the magnitude of w with respect to the system size N. Since the difference between $\hat{Z}$ and the average $\langle Z \rangle$ is expected to be small, we may regard w as being of the same order as the fluctuation of Z, i.e., $Z - \langle Z \rangle$. The key point is that the latter is $O(N^{-1/2})$ except at the critical point because correlation among the elements is finite-ranged, and hence the "central limit theorem" may be applied to the macroscopic quantity Z.[3] In fact, some numerical evidence has already been given for it (see Fig. 2 in ref. 16). Therefore, if we put

$$w = \tilde{w}/\sqrt{N} \tag{2.12}$$

$\tilde{w}$ is $O(1)$. [*Remark*. Our final results for $\langle |w|^2 \rangle$ derived later indeed reveal that w is $O(N^{-1/2})$.] Then the deviation of $\tilde{\theta}_j$ from ψ_j induced by w should also be of $O(N^{-1/2})$, so that we may expand $\tilde{\theta}_j$ in $N^{-1/2}$ as

$$\begin{aligned}\tilde{\theta}_j &= \psi_j + \phi_j \\ &= \psi_j + \frac{\tilde{\phi}_j}{\sqrt{N}} + O(N^{-1})\end{aligned} \tag{2.13}$$

Substituting (2.9), (2.12), and (2.13) into (2.1) to expand both sides of the latter in $N^{-1/2}$, we obtain

$$d\tilde{\phi}_j/dt = (\varepsilon/2\pi)\,\mathrm{Im}\{(-2\pi i'\hat{Z}\tilde{\phi}_j + \tilde{w})e^{-2\pi i'\psi_j}\} \tag{2.14}$$

by comparing $O(N^{-1/2})$ terms, where $\tilde{\phi}_j(0) = 0$. Furthermore, we may find from (1.3), (2.9), (2.12), and (2.13)

$$\tilde{w} = \sqrt{N}\left(-\hat{Z} + N^{-1}\sum_{j=1}^{N} e^{2\pi i'\psi_j}\right) + 2\pi i' N^{-1}\sum_{j=1}^{N} \tilde{\phi}_j e^{2\pi i'\psi_j} + O(N^{-1/2}) \tag{2.15}$$

The first term on the rhs of (2.15) is $O(1)$, in contrast to its appearance, since

$$\begin{aligned}-\hat{Z} + N^{-1}\sum_{j=1}^{N} e^{2\pi i'\psi_j} &\cong -\langle Z\rangle + N^{-1}\sum_{j=1}^{N} e^{2\pi i'\psi_j} \\ &\cong N^{-1}\sum_{j=1}^{N}(e^{2\pi i'\psi_j} - \langle e^{2\pi i'\psi_j}\rangle) = O(N^{-1/2})\end{aligned}$$

(Note that the ψ_j are independent of one another.) By inserting the solution to Eq. (2.14) into this equation and by taking the limit $N \to \infty$, we arrive at a self-consistent equation for $\tilde{w}$, which reads, in terms of the original variable w ($\tilde{w}$ will no longer be used, to avoid complexity in notation),

$$\begin{aligned}w(t) = &-\hat{Z} + N^{-1}\sum_{j=1}^{\cdot\cdot} e^{2\pi i'\psi_j(t)} \\ &+ \lambda\int_0^t dt'\,\{A_-(t,t')\,w(t') - A_+(t,t')\,w^*(t')\}\end{aligned} \tag{2.16}$$

[3] In the present case correlation among elements vanishes in the limit $N \to \infty$ because then we have a one-body problem due to the mean-field character of the evolution equations, as we have seen in the previous subsection. This fact may work very favorably for the application of the central limit theorem.

where $\lambda \equiv \varepsilon/2$, the asterisk means a complex conjugate, and the kernels $A_{\pm}$ are defined by

$$A_{\pm}(t, t') = \lim_{N \to \infty} N^{-1} \sum_{j=1}^{N} \exp\left\{2\pi i'[\psi_j(t) \pm \psi_j(t')] + \frac{\lambda}{\pi}\int_{t'}^{t} d\tau \, \mathrm{Im}(-2\pi i' \hat{Z} e^{-2\pi i' \psi_j(\tau)})\right\} \tag{2.17}$$

The linear integral equation for w derived above plays the most crucial role in the present paper. A careful review of the process of its derivation, however, may cast a doubt on the validity of the expansion in $N^{-1/2}$ invoked, because the solution to Eq. (2.14) may carry a secular part of $O(t)$ which limits the range of t where (2.13) is applicable to a finite interval. As a matter of fact, it is possible to overcome this difficulty by modifying the expansion so that the dominant phase motion absorbs such a dangerous part, and as a result, it turns out that all the outcome based on Eq. (2.16) remains valid. For convenience this is shown in Appendix B.

What remains to be done is to solve the linear integral equation for w, (2.16), for the subcritical and supercritical regimes separately, the solutions to which enable us to obtain all information on the behavior of fluctuations. In particular, the standard deviation σ and a correlation function may be evaluated from

$$\sigma = \lim_{N \to \infty} (N\langle |w - \langle w \rangle|^2 \rangle)^{1/2} \tag{2.18}$$

and

$$\begin{aligned} C_Z(\tau) &\equiv \lim_{N \to \infty} N\langle \{Z(t+\tau) - \langle Z \rangle\}\{Z(t) - \langle Z \rangle\}^* \rangle \\ &= \lim_{N \to \infty} N\langle \{w(t+\tau) - \langle w \rangle\}\{w(t) - \langle w \rangle\}^* \rangle \end{aligned} \tag{2.19}$$

respectively. We have thus completed the basic part of theory.

Our method to deal with fluctuations exploited above is based on a sort of system-size expansion, but it should be distinguished from van Kampen's method[30]: ours is concerned with purely deterministic systems, whereas the latter is for kinetic equations describing stochastic processes.

3. SUBCRITICAL REGIME

In this section we restrict ourselves to the disordered regime where the dominant phase motion is trivial:

$$\psi_j(t) = \Delta_j t + \theta_j(0) \tag{3.1}$$

by Eq. (2.11) with $\hat{Z}=0$, so that it is possible to derive exact expressions of σ and $C_Z(\tau)$. Let us first calculate the kernels: by (2.17) we have

$$A_-(t, t') = \lim_{N\to\infty} N^{-1} \sum_{j=1}^{N} e^{2\pi i'(t-t')\Delta_j}$$
$$= \int_{-\infty}^{\infty} d\Delta\, \mathcal{J}(\Delta) e^{2\pi i'(t-t')\Delta} \equiv A(t-t') \tag{3.2}$$

and

$$A_+(t, t') = \lim_{N\to\infty} N^{-1} \sum_{j=1}^{N} \exp\{2\pi i'[(t+t')\Delta_j + 2\theta_j(0)]\}$$
$$\equiv B(t+t') \tag{3.3}$$

which leads, via (2.16), to

$$w(t) = N^{-1} \sum_{j=1}^{N} \exp\{2\pi i'[\Delta_j t + \theta_j(0)]\}$$
$$+ \lambda \int_0^t dt' \, \{A(t-t')\, w(t') - B(t+t')\, w^*(t')\} \tag{3.4}$$

In order to solve Eq. (3.4), we need to note that the term accompanied by B may be neglected because it vanishes for $t\to\infty$. Then we obtain

$$w(t) = N^{-1} \sum_{j=1}^{N} \{1 - \lambda \tilde{A}(2\pi\Delta_j)\}^{-1} \exp\{2\pi i'[\Delta_j t + \theta_j(0)]\} \tag{3.5}$$

where

$$\tilde{A}(\omega) \equiv \int_0^{\infty} d\tau\, A(\tau) e^{-i'\omega\tau} \tag{3.6}$$

Equation (3.5) indicates that $w(t)$ exhibits quasiperiodic behavior on an N-torus. Making use of it and looking back at (2.18) and (2.19), we are able to reach the following results:

$$\langle |Z - \langle Z\rangle|^2\rangle \cong N^{-1} \int_{-\infty}^{\infty} d\Delta\, \mathcal{J}(\Delta)\, |1 - \lambda\hat{A}(2\pi\Delta)|^{-2} \tag{3.7}$$

so

$$\sigma = \left[\int_{-\infty}^{\infty} d\Delta\, \mathcal{J}(\Delta)\, |1 - \lambda\tilde{A}(2\pi\Delta)|^{-2}\right]^{1/2} \tag{3.8}$$

and

$$C_Z(\tau) = \int_{-\infty}^{\infty} d\Delta\, \tilde{f}(\Delta)\, |1 - \lambda \tilde{A}(2\pi\Delta)|^{-2} e^{2\pi i'\Delta\tau} \tag{3.9}$$

At this point it may be helpful to express $\tilde{A}$ by $\tilde{f}(\Delta)$: by (3.2) and (3.6)

$$\begin{aligned} \tilde{A}(2\pi\Delta) &= \lim_{\delta \to +0} \int_0^{\infty} d\tau\, A(\tau) e^{(-2\pi i'\Delta - \delta)\tau} \\ &= \frac{1}{2} \tilde{f}(\Delta) + \frac{i'}{2\pi} \mathrm{PV} \int_{-\infty}^{\infty} d\Delta'\, \tilde{f}(\Delta') \frac{1}{\Delta' - \Delta} \end{aligned} \tag{3.10}$$

where PV means a principal value integral. In this way we have found a pair of formulas for the intensity of fluctuations as well as the correlation function as ε-dependent functionals of the native frequency distribution.

Example. For a Lorentzian

$$\tilde{f}(\Delta) = (\gamma/\pi)(\Delta^2 + \gamma^2)^{-1} \tag{3.11}$$

we have

$$\sigma = \left(\frac{4\pi\gamma}{4\pi\gamma - \varepsilon} \right)^{1/2} \tag{3.12}$$

and

$$C_Z(\tau) = \frac{4\pi\gamma}{4\pi\gamma - \varepsilon} \exp\left\{ -\left(\frac{4\pi\gamma - \varepsilon}{2} \right) \tau \right\} \tag{3.13}$$

where $\varepsilon_c = 4\pi\gamma$ by (2.7).

Let us now proceed to check the critical behavior of σ. For simplicity we assume that the symmetric distribution $\tilde{f}(\Delta)$ is monotonic on both sides of $\Delta = 0$. Then it follows from (3.10) and (2.7) that the denominator of the integrand in (3.7), $1 - \lambda \tilde{A}(2\pi\Delta)$, never becomes zero for $\varepsilon < \varepsilon_c$, vanishing only at $\Delta = 0$ for $\varepsilon = \varepsilon_c$. Moreover, it is easy to see for $\varepsilon = \varepsilon_c(1 - \delta)$ with $0 < \delta \ll 1$ and for small Δ that

$$1 - \lambda \tilde{A}(2\pi\Delta) = \delta - i'(\varepsilon_c b/2)\Delta + O(\Delta^2) + O(\delta\Delta) \tag{3.14}$$

where

$$b \equiv \frac{1}{\pi} \int_0^{\infty} dy \frac{\tilde{f}'(y)}{y} < 0$$

By way of (3.8) these observations lead to

$$\sigma \cong \left[\frac{2\pi \tilde{f}(0)}{|b|}\right]^{1/2} (\varepsilon_c - \varepsilon)^{-1/2} \tag{3.15}$$

for $\varepsilon \to \varepsilon_c^-$. Namely, it turns out that γ' equals 1/2 for the subcritical regime [see (1.8)].

We now examine the critical behavior of the correlation function. The long-time behavior of $C_Z(\tau)$ is dominated near ε_c by a pole of the integrand in (3.9), which may be calculated from (3.14) as

$$\Delta = i' \frac{2}{|b|\,\varepsilon_c} \delta + O(\delta^2) \equiv i'q \tag{3.16}$$

so that the correlation of fluctuations damps exponentially as

$$C_Z(\tau) \cong \frac{2\pi \tilde{f}(0)}{|b|} (\varepsilon_c - \varepsilon)^{-1} e^{-2\pi q \tau} \tag{3.17}$$

for $\tau \gg 1$. The decay time or the correlation time τ_c may be given by

$$\begin{aligned} \tau_c &= (2\pi \operatorname{Re} q)^{-1} \\ &\cong \frac{4\,|b|}{\pi \tilde{f}(0)^2} (\varepsilon_c - \varepsilon)^{-1} \end{aligned} \tag{3.18}$$

which indicates the occurrence of critical slowing down at the onset of mutual entrainment.

Equations (3.12) and (3.13) clearly show that the critical singularities deduced above indeed appear for the Lorentzian distribution (3.11). [Curiously enough, the critical scalings exactly hold everywhere in the disordered regime for such an $\tilde{f}(\Delta)$.] In this way we have shown analytically that critical divergence of σ as well as τ_c takes place as the coupling strength approaches its critical value from below. It is important to note that the critical exponents are just the same as those for thermal phase transitions in systems with mean-field type interactions.[25]

4. SUPERCRITICAL REGIME

In the macroscopically ordered phase the dominant phase motion may be separated into two types because of the nonvanishing order parameter, i.e., each ψ is either entrained or nonentrained [see (2.2) and (2.4)]. When entrained, ψ_j stays at the constant $\tilde{\theta}_j^*$ in (2.3) and its contribution to the kernels $A_\pm$ can be easily evaluated with rigor. Otherwise, it behaves in a nontrivial way, though periodic (with mod. 1), rendering exact calculation of the kernels difficult. Therefore, in what follows, a certain reasonable approximation will be employed for $A_\pm$, unlike in the previous section for the subcritical regime.

4.1. Evaluation of the Kernels

Let us begin by rewriting $A_\pm$ as follows:

$$A_-(t, t') = \lim_{N \to \infty} N^{-1} \sum_{j=1}^{N} e^{X_j(t) - X_j(t')} \tag{4.1}$$

and

$$A_+(t, t') = \lim_{N \to \infty} N^{-1} \sum_{j=1}^{N} e^{X_j(t) - X_j^*(t')} \tag{4.2}$$

where

$$X_j(t) \equiv 2\pi i' \psi_j(t) + (\lambda/\pi) \int_0^t d\tau \, \mathrm{Im}\{-2\pi i' \hat{Z} e^{-2\pi i' \psi_j(\tau)}\} \tag{4.3}$$

The contribution by the group of entrained oscillators [see (2.2)] $A_\pm^{(e)}$ may be expressed by putting $\psi_j = \tilde{\theta}_j^*$ as

$$\begin{aligned} A_-(t, t')^{(e)} &= \int_{|\Delta| < Q} d\Delta \, \tilde{f}(\Delta) \exp\{-2\pi(Q^2 - \Delta^2)^{1/2}(t - t')\} \\ &\equiv C_e(t - t') \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} A_+(t, t')^{(e)} = e^{4\pi i' \alpha} \int_{|\Delta| < Q} d\Delta \, \tilde{f}(\Delta) \left(1 - \frac{2\Delta^2}{Q^2}\right) \\ \times \exp\{-2\pi(Q^2 - \Delta^2)^{1/2}(t - t')\} \equiv D_e(t - t') \end{aligned} \tag{4.5}$$

where Q is defined by

$$Q = (\varepsilon/2\pi) \, |\hat{Z}| = (\lambda/\pi) |\hat{Z}| \tag{4.6}$$

In order to estimate of the remaining part of $A_\pm$, we invoke an approximation which is based on the fact that if we put

$$X_j(t) = 2\pi i' \psi_j(0) + 2\pi i' \, \mathrm{sgn}(\Delta_j)(\Delta_j^2 - Q^2)^{1/2} t + \hat{X}_j(t) \tag{4.7}$$

for a nonentrained oscillator, the last term $\hat{X}_j(t)$ is periodic with period T_j given by (2.5), as is proved using

$$\langle e^{2\pi i'\psi_j}\rangle = T_j^{-1}\int_0^{T_j} d\tau\, e^{2\pi i'\psi_j(\tau)}$$
$$= i'\,\mathrm{sgn}(\Delta_j)\,\frac{(\lambda/\pi)\hat{Z}}{(\Delta_j^2-Q^2)^{1/2}+|\Delta_j|} \tag{4.8}$$

For $t \gg 1$ the bounded variation of $\hat{X}_j(t)$ should be small compared to the rest of $X_j(t)$, so that we discard it and put

$$X_j(t) \to 2\pi i'\psi_j(0) + 2\pi i'\,\mathrm{sgn}(\Delta_j)(\Delta_j^2-Q^2)^{1/2}t \tag{4.9}$$

for nonentrained oscillators. In this way $A_\pm^{(\mathrm{ne})} \equiv A_\pm - A_\pm^{(\mathrm{e})}$ is approximately given by

$$A_-(t,t')^{(\mathrm{ne})} = \int_{|\Delta|>Q} d\Delta\, \tilde{f}(\Delta)\exp\{2\pi i'\,\mathrm{sgn}(\Delta)(\Delta^2-Q^2)^{1/2}(t-t')\}$$
$$\equiv C_{\mathrm{ne}}(t-t') \tag{4.10}$$

and

$$A_+(t,t')^{(\mathrm{ne})} = \lim_{N\to\infty} N^{-1}\sum_j{}' \exp\{4\pi i'\psi_j(0) + 2\pi i'\,\mathrm{sgn}(\Delta_j)(\Delta_j^2-Q^2)^{1/2}(t+t')\}$$
$$\equiv D_{\mathrm{ne}}(t+t') \tag{4.11}$$

where $\sum_j'$ stands for a summation over nonentrained oscillators.

To sum up, the kernels are given by

$$A_-(t,t') = C_{\mathrm{e}}(t-t') + C_{\mathrm{ne}}(t-t') \equiv C(t-t') \tag{4.12}$$

and

$$A_+(t,t') = D_{\mathrm{e}}(t-t') + D_{\mathrm{ne}}(t+t') \equiv D(t,t') \tag{4.13}$$

As to the approximation (4.9), note that by (4.3), (4.7), and (2.11), the periodic function $\hat{X}_j(t)$ disappears for $\varepsilon \doteq \varepsilon_c$. This may suggest that the approximation is especially efficient near the critical point.

4.2. Solving the Integral Equation for w

Given the explicit forms of the kernels, we proceed to look for a solution to Eq. (2.16), which now reads

$$w(t) = w_0 + N^{-1}\sum_j{}' \{e^{2\pi i'\psi_j(t)} - \langle e^{2\pi i'\psi_j}\rangle\}$$
$$+ \lambda\int_0^t dt'\, \{C(t-t')\,w(t') - D(t,t')\,w^*(t')\} \tag{4.14}$$

where w_0 is a constant defined by

$$w_0 = -\hat{Z} + N^{-1} \sum_{j=1}^{N} \langle e^{2\pi i' \psi_j} \rangle \tag{4.15}$$

Let us estimate the magnitude of w_0 for N large. Making use of (2.3) and (4.8), and noting (2.6), we obtain from (4.15)

$$\begin{aligned} e^{-2\pi i' \alpha} w_0 &= -|\hat{Z}| + N^{-1} \sum_{|\Delta_j| < Q} [1 - (\Delta_j/Q)^2]^{1/2} \\ &= -2 \int_0^Q d\Delta\, \tilde{f}(\Delta)[1 - (\Delta/Q)^2]^{1/2} + N^{-1} \sum_{|\Delta_j| < Q} [1 - (\Delta_j/Q)^2]^{1/2} \end{aligned}$$

For simplicity let us suppose that the frequencies of entrained oscillators are numbered $\Delta_0 = 0 < \Delta_1 < \Delta_2 < \cdots < \Delta_n = Q$ for $\Delta \geqslant 0$, and then, by symmetry, those for $\Delta < 0$ may be given by $-\Delta_k$ $(1 \leqslant k \leqslant n)$. The above equation leads to

$$\begin{aligned} e^{-2\pi i' \alpha} w &= N^{-1} + 2N^{-1} \sum_{k=1}^{n} [1 - (\Delta_k/Q)^2]^{1/2} \\ &\quad -2 \sum_{k=1}^{n} \int_{\Delta_{k-1}}^{\Delta_k} d\Delta\, \tilde{f}(\Delta)[1 - (\Delta/Q)^2]^{1/2} \end{aligned}$$

which can be shown to be $O(N^{-1})$, noting that we may put

$$\begin{aligned} N^{-1} &= \int_{\Delta_{k-1}}^{\Delta_k} d\Delta\, \tilde{f}(\Delta) \\ &= \tilde{f}(\Delta_{k-1})(\Delta_k - \Delta_{k-1}) + \cdots \qquad (1 \leqslant k \leqslant n) \end{aligned}$$

for N large. Therefore, we will omit w_0 in (4.14) hereafter. It is convenient to cast w and all of $e^{2\pi i' \psi}$ into the form of the Fourier representation:

$$w(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, \tilde{w}(\omega) e^{i' \omega t} \tag{4.16}$$

and

$$e^{2\pi i' \psi_j(t)} - \langle e^{2\pi i' \psi_j} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, \tilde{\psi}_j(\omega) e^{i' \omega t} \tag{4.17}$$

by which we obtain from (4.14)

$$\begin{aligned} \tilde{w}(\omega) &= N^{-1} \sum_j{}' \tilde{\psi}_j(\omega) \\ &\quad + \lambda \tilde{C}(\omega)\, \tilde{w}(\omega) - \lambda \tilde{D}_e(\omega)\, \tilde{w}^*(-\omega) \end{aligned} \tag{4.18}$$

where

$$\tilde{C}(\omega) \equiv \int_0^\infty d\tau\, C(\tau) e^{-i'\omega\tau} \tag{4.19}$$

and

$$\tilde{D}_e(\omega) \equiv \int_0^\infty d\tau\, D_e(\tau) e^{-i'\omega\tau} \equiv e^{4\pi i'\alpha} \tilde{U}(\omega) \tag{4.20}$$

Note that $D_{ne}(t+t')$ does not affect the asymptotic behavior of $w(t)$ for $t \gg 1$ just like $B(t+t')$ in the subcritical regime [see (3.3)]. Solving Eq. (4.18), we finally arrive at

$$\begin{aligned} w(t) = N^{-1} {\sum_j}' \sum_{\substack{m=-\infty \\ (\neq 0)}}^{\infty} &\{d_m(\Delta_j)\, P(2\pi m(\Delta_j^2 - Q^2)^{1/2}) \\ &- d^*_{-m}(\Delta_j)\, R(2\pi m(\Delta_j^2 - Q^2)^{1/2})\} \\ &\times \exp[2\pi i' m(\Delta_j^2 - Q^2)^{1/2} t] \end{aligned} \tag{4.21}$$

where

$$P(\omega) \equiv \frac{1 - \lambda\tilde{C}(\omega)}{\{1 - \lambda\tilde{C}(\omega)\}^2 - \lambda^2 \tilde{U}(\omega)^2} \tag{4.22}$$

$$R(\omega) \equiv \frac{e^{4\pi i'\alpha} \lambda \tilde{U}(\omega)}{\{1 - \lambda\tilde{C}(\omega)\}^2 - \lambda^2 \tilde{U}(\omega)^2} \equiv e^{4\pi i'\alpha} S(\omega) \tag{4.23}$$

and for convenience we have introduced $d_m(\Delta_j)$ by

$$e^{2\pi i'\psi_j(t)} = \sum_{m=-\infty}^{\infty} d_m(\Delta_j) e^{2\pi i' m t/T_j} \tag{4.24}$$

In this way we see that in the supercritical regime $w(t)$ performs multiple-periodic motion just as on the opposite side of the threshold. There is an important difference between the motions of $w(t)$ in both regimes: the dimensionality of the multiple periodicity or the torus N_{ne} depends on the parameter in the ordered phase as

$$N_{ne} = N - N \int_{|\Delta| < Q} d\Delta\, \tilde{f}(\Delta) \tag{4.25}$$

unlike in the disordered phase, where it never ceases to be N. As we shall see later, this distinction results in some remarkable differences of the fluctuation property between both regimes.

Before going on to the next subsection, let us make some manipulations of the Fourier transforms $\tilde{C}$ and $\tilde{U}$ for later use. By (4.19) as well as (4.4) and (4.10) we have

$$\tilde{C}(\omega) = \tilde{C}_e(\omega) + \tilde{C}_{ne}(\omega) \tag{4.26}$$

where

$$\begin{aligned}\tilde{C}_e(\omega) &\equiv \int_0^\infty d\tau\, e^{-i'\omega\tau} \int_{|\Delta|<Q} d\Delta\, \tilde{f}(\Delta) \exp[-2\pi\tau(Q^2-\Delta^2)^{1/2}] \\ &= Q \int_{|x|<1} dx\, \tilde{f}(Qx) \frac{1}{2\pi Q(1-x^2)^{1/2} + i'\omega}\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}\tilde{C}_{ne}(\omega) &\equiv \int_0^\infty d\tau\, e^{-i'\omega\tau} \int_{|\Delta|>Q} d\Delta\, \tilde{f}(\Delta) \exp[2\pi i' \operatorname{sgn}(\Delta)\, \tau(\Delta^2-Q^2)^{1/2}] \\ &= \lim_{\delta\to+0} \int_{|\Delta|>Q} d\Delta\, \tilde{f}(\Delta) \int_0^\infty d\tau \\ &\quad \times \exp\{[-\delta - i'\omega + 2\pi i' \operatorname{sgn}(\Delta)(\Delta^2-Q^2)^{1/2}]\tau\} \\ &= \frac{1}{2}\frac{|\tilde{\omega}|}{(1+\tilde{\omega}^2)^{1/2}} \tilde{f}\,(Q(1+\tilde{\omega}^2)^{1/2}) \\ &\quad + \frac{i'}{2\pi} \mathrm{PV} \int_{-\infty}^\infty dy \frac{|y|}{(1+y^2)^{1/2}} \tilde{f}(Q(1+y^2)^{1/2}) \frac{1}{y-\tilde{\omega}}\end{aligned} \tag{4.28}$$

For $\tilde{U}$, we find from (4.5) and (4.20)

$$\tilde{U}(\omega) = Q \int_{|x|<1} dx\, \tilde{f}(Qx) \frac{1-2x^2}{2\pi Q(1-x^2)^{1/2} + i'\omega} \tag{4.29}$$

4.3. The Intensity of Fluctuations

According to (4.21), we have

$$\langle |Z - \langle Z\rangle|^2\rangle \cong N^{-1} F \tag{4.30}$$

that is,

$$\sigma = \sqrt{F} \tag{4.31}$$

where

$$\begin{aligned}F &\equiv \int_{|\Delta|>Q} d\Delta\, \tilde{f}(\Delta) \sum_{\substack{m=-\infty \\ (\neq 0)}}^{\infty} \\ &\quad \times |Y_m(\Delta)\, P(2\pi m(\Delta^2-Q^2)^{1/2}) - Y^*_{-m}(\Delta)\, S(2\pi m(\Delta^2-Q^2)^{1/2})|^2\end{aligned} \tag{4.32}$$

See Appendix A for $Y_m(\Delta)$, which is defined in (A.3) so as to have no dependence on initial conditions of our oscillator system, and is related to $d_m(\Delta)$ as in (A.6). In order to examine the critical behavior of σ, we focus on the analysis of the factor F below.

Let us first note that the sum in (4.32) is an even function of Δ, as is easily proved using

$$P^*(-\omega) = P(\omega), \qquad S^*(-\omega) = S(\omega) \tag{4.33}$$

as well as (A.4) in Appendix A. Moreover, it is useful to notice that Y_m can be expressed with an ε-independent function $\hat{Y}_m$ as

$$Y_m(\Delta) = \hat{Y}_m(\Delta/Q) \tag{4.34}$$

as will be justified in Appendix A2. Then some manipulations lead from (4.32) to

$$F/2 = \sum_{\substack{m=-\infty \\ (\neq 0)}}^{\infty} (F_m + F_m' + F_m'') \tag{4.35}$$

with

$$\begin{aligned} F_m &= \int_0^{Qc_m} dx\, \Phi_m(x) \\ F_m' &= \int_{Qc_m}^{c_m} dx\, \Phi_m(x) \\ F_m'' &= \int_{c_m}^{\infty} dx\, \Phi_m(x) \end{aligned} \tag{4.36}$$

where

$$\begin{aligned} \Phi_m(x) \equiv \frac{x}{(x^2+Q^2)^{1/2}} \tilde{J}\left((x^2+Q^2)^{1/2}\right) \Bigg| \hat{Y}_m\left(\left[\left(\frac{x}{Q}\right)^2+1\right]^{1/2}\right) P(2\pi m x) \\ - \hat{Y}^*_{-m}\left(\left[\left(\frac{x}{Q}\right)^2+1\right]^{1/2}\right) S(2\pi m x)\Bigg|^2 \end{aligned} \tag{4.37}$$

and c_m is a positive constant of the order $1/|m|$. In what follows we examine the critical behavior of $\sum F_m$, $\sum F_m'$, and $\sum F_m''$ for $\varepsilon \to \varepsilon_c^+$ to find out how σ scales at the onset of mutual entrainment.

(a) Analysis of $\sum F_m$. Let us start by rewriting F_m in (4.36) as

$$\begin{aligned} F_m = Q \int_0^{c_m} dy \frac{y}{(1+y^2)^{1/2}} \tilde{J}(Q(1+y^2)^{1/2}) \,|\, \hat{Y}_m((1+y^2)^{1/2})\, P(2\pi Q m y) \\ - \hat{Y}^*_{-m}((1+y^2)^{1/2})\, S(2\pi Q m y)|^2 \end{aligned} \tag{4.38}$$

From the definition of c_m, we may estimate this integral by retaining the dominant term alone in the expansion of the integrand in both y and my. For this purpose we quote the following result of Appendix A3: for small y

$$\hat{Y}_m((1+y^2)^{1/2}) = i'(\mathscr{D} + a_m)\, y + o(y) \tag{4.39}$$

where

$$a_m \equiv (-1)^m (2/\pi) \int_0^{\pi/2} dx\, \sin(2mx) \tan x$$

and $\mathscr{D}$ is a constant. The expressions for $P(2\pi Qz)$ and $S(2\pi Qz)$ for small z may be obtained via (4.22) and (4.23) from the following:

$$1 - \lambda\tilde{C}(2\pi Qz) = 1 - (\lambda/2\pi) \int_{|x|<1} dx\, \frac{\tilde{J}(Qx)}{(1-x^2)^{1/2}} + (\cdots)z + o(z) \tag{4.40}$$

and

$$\lambda\tilde{U}(2\pi Qz) = (\lambda/2\pi) \int_{|x|<1} dx\, \frac{\tilde{J}(Qx)(1-2x^2)}{(1-x^2)^{1/2}} + (B_1 + \cdots) z \ln|z| + (B_2 + \cdots) z + o(z) \tag{4.41}$$

where

$$B_1 \equiv -i' \frac{2}{\pi}, \qquad B_2 \equiv \operatorname{sgn}(z) - i' \frac{2}{\pi} (2 - \ln 2)$$

and "$\cdots$" stands for the part of the coefficient vanishing for $\varepsilon \to \varepsilon_c$. Both of the above formulas are derived in Appendix A4. It should also be noted that by (2.6)

$$1 - \lambda\tilde{C}(0) = \lambda\tilde{U}(0) \equiv A_0 \propto Q^2 \qquad (\varepsilon \to \varepsilon_c) \tag{4.42}$$

With (4.38)–(4.42), F_m may be estimated near ε_c as

$$\frac{F_m}{Q} \cong \frac{\tilde{J}(0)}{m^4} \left| -a_m + \mathscr{D}^* \right|^2 \int_0^{mc_m} dx\, \frac{x}{|B_1 \ln|x| + B_2|^2} \tag{4.43}$$

which is $O(m^{-4})$, so that we have

$$\sum_{\substack{m=-\infty \\ (\neq 0)}}^{\infty} F_m = O(Q) \tag{4.44}$$

(b) Analysis of $\sum F'_m$. By definition

$$F'_m = \int_{Qc_m}^{c_m} dx \frac{x}{(x^2+Q^2)^{1/2}} \tilde{J}((x^2+Q^2)^{1/2}) \times \left| \hat{Y}_m\left(\left[\left(\frac{x}{Q}\right)^2+1\right]^{1/2}\right) P(2\pi mx) - \hat{Y}^*_{-m}\left(\left[\left(\frac{x}{Q}\right)^2+1\right]^{1/2}\right) S(2\pi mx)\right|^2 \tag{4.45}$$

When $|m| \ll Q^{-1}$, the argument of $\hat{Y}_m$ is typically large in the range of integration, so that we are allowed to invoke the asymptotic forms of $\hat{Y}_m(x)$ for $x \gg 1$ worked out in Appendix A5:

$$\hat{Y}_m(x) \cong 1 \quad (m=1), \qquad 1/2(1-m)x^2 \quad (m \neq 1) \tag{4.46}$$

This observation may suggest dividing the sum in such a way that

$$\sum_{\substack{m=-\infty \\ (\neq 0)}}^{\infty} F'_m = F'_1 + F'_{-1} + \sum_{1<|m|\leq m_1} F'_m + \sum_{|m|>m_1} F'_m \tag{4.47}$$

where m_1 is an integer of $O(Q^{-1})$. First, by (4.46), we have

$$F'_1 \cong \int_{Qc_1}^{c_1} dx \frac{x}{(x^2+Q^2)^{1/2}} \tilde{J}((x^2+Q^2)^{1/2}) \, |P(2\pi x)|^2$$

$$\cong \tilde{J}(0) \int_{Qc_1}^{c_1} dx \, |P(2\pi x)|^2 \tag{4.48}$$

and in the same way

$$F'_{-1} \cong \tilde{J}(0) \int_{Qc_{-1}}^{c_{-1}} dx \, |S(2\pi x)|^2 \tag{4.49}$$

where (4.33) has been used. Second, looking again at (4.46), we obtain for m as $1 < |m| \leq m_1$

$$F'_m \cong \int_{Qc_m}^{c_m} dx \frac{x}{(x^2+Q^2)^{1/2}} \tilde{J}((x^2+Q^2)^{1/2}) \times \frac{1}{4\{(x/Q)^2+1\}^2} \left| \frac{1}{m-1} P(2\pi mx) + \frac{1}{m+1} S(2\pi mx)\right|^2$$

$$< \frac{\tilde{J}(0)}{4} e_m^2 \int_{Qc_m}^{c_m} dx \frac{\{|P(2\pi mx)| + |S(2\pi mx)|\}^2}{\{(x/Q)^2+1\}^2} \tag{4.50}$$

where $e_m \equiv \mathrm{Max}\{1/|m-1|,\ 1/|m+1|\}$. Finally, when $|m| > m_1$, we use (4.39) to find

$$F'_m \cong \tilde{J}(0) Q^{-3} \int_{Qc_m}^{c_m} dx\, x^3\, |\mathscr{D} P(2\pi m x) + \mathscr{D}^* S(2\pi m x)|^2 \tag{4.51}$$

where a_m has been discarded due to its smallness for large m.

Let us now discuss the critical behavior of the above expressions. It is convenient to note that if we put $mx = \sqrt{Q}\, z$, we have

$$\begin{aligned}&[|P(2\pi m x)| + |S(2\pi m x)|]^2 \\ &\qquad \cong \frac{\pi^2}{Q} \frac{z^2}{(1/3 + b' z^2)^2} \equiv \frac{\pi^2}{Q} V(z)\end{aligned} \tag{4.52}$$

for $Q \to 0$, where

$$b' \equiv -2\tilde{J}(0)^{-1} \int_0^\infty dy\, \tilde{J}'(y)/y > 0$$

and the following results in Appendix A6 have been used:

$$1 - \lambda \tilde{C}(2\pi\sqrt{Q}\, z) = \frac{i'\sqrt{Q}}{\pi} \left(\frac{1}{z} + b' z\right) + o(\sqrt{Q}) \tag{4.53}$$

$$\lambda \tilde{U}(2\pi\sqrt{Q}\, z) = -\frac{2i'\sqrt{Q}}{3\pi z} + o(\sqrt{Q}) \tag{4.54}$$

First let us consider the sum of F'_m over $1 < |m| \leqslant m_1$, whose upper bound may be estimated as follows [see (4.50) and (4.52)]:

$$\begin{aligned}
&\sum_{1<|m|\leqslant m_1} e_m^2 \int_{Qc_m}^{c_m} dx \frac{\{|P(2\pi m x)| + |S(2\pi m x)|\}^2}{\{(x/Q)^2 + 1\}^2} \\
&\quad \cong \sum_{1<|m|\leqslant m_1} e_m^2 \int_{Qc_m}^{Q} dx\, \{|P(2\pi m x)| + |S(2\pi m x)|\}^2 \\
&\qquad + \sum_{1<|m|\leqslant m_1} e_m^2 \int_{Q}^{c_m} dx \left(\frac{Q}{x}\right)^4 \{|P(2\pi m x)| + |S(2\pi m x)|\}^2 \\
&\quad \cong \sum_{1<|m|\leqslant m_1} \pi^2 m^{-1} e_m^2 Q^{-1/2} \int_{(mc_m)\sqrt{Q}}^{m\sqrt{Q}} dz\, V(z) \\
&\qquad + \sum_{1<|m|\leqslant m_1} \pi^2 m^3 e_m^2 Q^{3/2} \int_{m\sqrt{Q}}^{(mc_m)/\sqrt{Q}} dz\, V(z) z^{-4} \\
&\quad = O(\sqrt{Q})
\end{aligned} \tag{4.55}$$

Therefore we see that the sum concerned is at most $Q(\sqrt{Q})$. It is also possible to show on the basis of (4.51) that the sum of F'_m over $|m| > m_1$ is at most $O(1)$. Finally, using (4.53) and (4.54), we find from (4.48) and (4.49)

$$F'_1 + F'_{-1} \cong \frac{\pi^2 \tilde{J}(0)}{\sqrt{Q}} \int_0^\infty dz \frac{z^2\{(1+b'z^2)^2 + 4/9\}}{(1/3 + b'z^2)^2(5/3 + b'z^2)^2}$$
$$\propto Q^{-1/2} \tag{4.56}$$

These "lemmas" indicate together with (4.47) a curious singular behavior of the original sum near ε_c as

$$\sum_{\substack{m=-\infty \\ (\neq 0)}}^{\infty} F'_m \propto (\varepsilon - \varepsilon_c)^{-1/4} \tag{4.57}$$

(c) Analysis of $\sum F''_m$. It is necessary to note that for $\varepsilon = \varepsilon_c$, $F''_1 < \infty$ as well as $F''_m = 0$ $(m \neq 1)$, which is shown on the basis of (4.46) and the fact that, by (4.27)–(4.29), $\tilde{C}(\omega) \equiv \tilde{A}(\omega)$ and $\tilde{U}(\omega) \equiv 0$ just at the threshold. Therefore we simply expect that for $\varepsilon \to \varepsilon_c$

$$\sum_{\substack{m=-\infty \\ (\neq 0)}}^{\infty} F''_m < \infty \tag{4.58}$$

Looking back at the results of (a)–(c), i.e., (4.44), (4.57), and (4.58), and also at (4.35) and (4.31), we now arrive at the critical scaling law for σ:

$$\sigma \propto (\varepsilon - \varepsilon_c)^{-1/8} \tag{4.59}$$

In other words, $\gamma' = 1/8$ in the macroscopically ordered phase, which is surprisingly small in comparison with the subcritical exponent 1/2.

4.4. Correlation Function of *Z* and Its Critical Behavior

By (4.21) and (2.19) the correlation function may be given by

$$C_Z(\tau) = \int_{|\Delta| > Q} d\Delta\, \tilde{J}(\Delta) \sum_{\substack{m=-\infty \\ (\neq 0)}}^{\infty} |d_m(\Delta)\, P(2\pi m(\Delta^2 - Q^2)^{1/2})$$
$$- d^*_{-m}(\Delta)\, R(2\pi m(\Delta^2 - Q^2)^{1/2})|^2 \exp[2\pi i' m(\Delta^2 - Q^2)^{1/2} \tau]$$
$$= 2 \sum_{\substack{m=-\infty \\ (\neq 0)}}^{\infty} \int_0^\infty dx\, \Phi_m(x) \cos(2\pi m x \tau) \tag{4.60}$$

According to the preceding subsection, we have near ε_c

$$\begin{aligned} C_Z(\tau) &\cong 2 \sum_{\substack{m=-\infty \\ (\neq 0)}}^{\infty} \int_{Qc_m}^{c_m} dx\, \Phi_m(x) \cos(2\pi m x \tau) \\ &\cong \frac{2\pi^2 \tilde{f}(0)}{\sqrt{Q}} \int_0^\infty dz\, \frac{z^2\{(1+b'z^2)^2 + 4/9\}}{(1/3+b'z^2)^2(5/3+b'z^2)^2} \cos(2\pi\sqrt{Q}\,\tau z) \\ &= \frac{\pi^3\sqrt{3}\,\tilde{f}(0)}{4(b')^{3/2}\sqrt{Q}}\, \Pi(\sqrt{Q}\,\tau) \end{aligned} \tag{4.61}$$

where

$$\Pi(x) \equiv e^{-sx} + \frac{e^{-\sqrt{5}\,sx}}{\sqrt{5}} - sx(e^{-sx} + e^{-\sqrt{5}\,sx})$$

with $s \equiv 2\pi/(3b')^{1/2}$. Critical slowing down thus takes place in such a way that

$$\tau_c = (s\sqrt{Q})^{-1} \propto (\varepsilon - \varepsilon_c)^{-1/4} \tag{4.62}$$

in which the critical index again disagrees with the typical mean-field value discovered for the subcritical regime [compare with (3.18)].

In this section we have given a quantitative description of the statistical behavior of $Z(t)$ for the supercritical regime on the basis of the framework developed earlier with the approximation (4.9) employed. In particular, it has been shown that the intensity as well as the characteristic time of fluctuations exhibit critical divergence at the onset of entrainment in this regime as well, but with indices much less than the familiar values found in the nonentrained phase. This peculiar discrepancy of the exponents may bring the uniqueness of our nonequilibrium phase transition into relief when compared with conventional ones, whose origin will be discussed in Section 6 after checking some results of numerical simulations in Section 5.

5. FINITE-SIZE SCALING AND NUMERICAL EVIDENCE

In numerical computations as well as laboratory experiments, finite-size effects are expected to become more and more serious as the threshold is approached, causing a significant discrepancy of the critical behaviors from those in the thermodynamic limit. It is thus indispensable to exploit finite-size scaling as a remedy for this problem, which is a well-established technique in equilibrium cooperative phenomena.[31] In this section we undertake to verify the theoretical results presented so far, mainly with the aid of finite-size scaling analyses.

In previous work,[16] σ was demonstrated to obey a supercritical finite-size scaling[4]

$$\sigma = N^{1/4}\Psi_+(N^2(\varepsilon - \varepsilon_c)) \tag{5.1}$$

for ε near ε_c and for large N, where $\Psi_+(x)$ is a universal function such that $\Psi_+(x) \sim x^{-1/8}$ for $x \gg 1$. (*Remark*: σ in ref. 16 is $\sigma/\sqrt{N}$ in the present notation.) The scaling law implies that finite-size effects become appreciable when the distance from ε_c is as small as $O(N^{-2})$. This may be understood intuitively as follows: the key is the number of entrained oscillators, which has to be sufficiently large to assure critical scalings for $N \to \infty$, i.e.,

$$N \int_{|\Delta| < Q} d\Delta\, \tilde{f}(\Delta) \sim 2N\tilde{f}(0)Q \gg 1 \tag{5.2}$$

This explanation is essentially the same as the one proposed in ref. 16 on the basis of the order parameter behavior. Once the scaling (5.1) is admitted, it is straightforward to predict the subcritical scaling law: by continuity of σ at the critical point it should be of the form

$$\sigma = N^{1/4}\Psi_-(N^{1/2}(\varepsilon_c - \varepsilon)) \tag{5.3}$$

where $\Psi_-(x)$ must decay as $x^{-1/2}$ for $x \gg 1$ to be consistent with (3.15). Namely, finite-size effects should work substantially in the regime $|\varepsilon - \varepsilon_c| \leqslant O(N^{-1/2})$ in the disordered phase. From now on let us express the size of a critical regime of this type by $O(N^{-s})$ and call s a crossover exponent. For $\varepsilon < \varepsilon_c$ it is equivalent to that of classical spin systems with uniform infinite-range interactions,[32] while this is not so for $\varepsilon > \varepsilon_c$. We thus have another example of the discrepancy between subcritical and supercritical exponents. Although it is a direct consequence of the same feature of γ', it may deserve special attention because of its phenomenological importance. In the next section a clear-cut physical picture will be given of the crossover phenomenon.

We are now ready to survey what has been observed by numerical simulations. As in previous work, the discrete-time model, (1.6), was used for a Lorentzian distribution of Ω_j with $\gamma = 0.001$ [see (3.11)] to approximate the model equation (1.2). (Refer to previous papers[15,16] for how to realize the distribution.) All the computations were carried out for fairly random initial conditions with first a few thousand iterations discarded as a transient regime. Figure 3 shows the behavior of σ in a global

[4] σ in this section stands for the rhs of (2.18) with the limit $N \to \infty$ not taken.

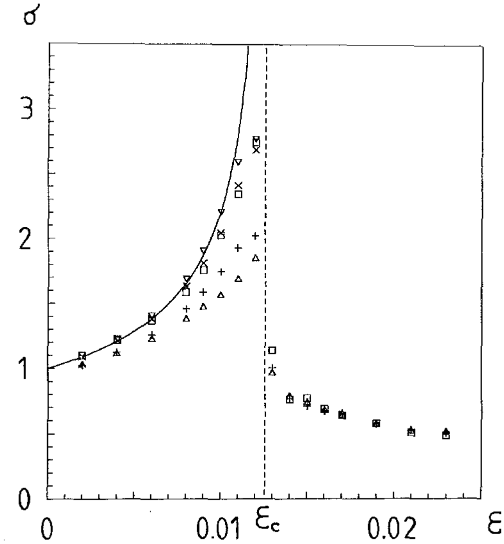

Fig. 3. Plot of σ vs. ε. Numerical data were obtained by averaging over 2^{15} iterations for $N = 1600$ ($\triangle$), 3000 (+), 6000 ($\square$), 10,000 ($\times$), and 20,000 (∇). (For $\varepsilon > \varepsilon_c$ the largest N is 6000.) The curve shows the theoretical result, (3.12), while the broken line locates the threshold of the phase transition given by (2.7).

range of the parameter for a variety of system sizes. Figure 4 shows finite-size scaling plots to test (5.1) and (5.3).

Let us first concentrate on the subcritical region. It is evident in Fig. 3 that even $N = 3000$ is still too small to capture the behavior of σ in the thermodynamic limit, part of which begins to appear for $N \sim 6000$. The theory, (3.12), is seen to be in good agreement with the data for larger N, except a vicinity of the critical point. In Fig. 4a it is demonstrated that the intensity of fluctuations indeed follows the scaling law (5.3). [*Remark.* In ref. 16 the small values of $N(\leqslant 1600)$ resulted in the erroneous conclusion that γ' is close to 1/8 even for $\varepsilon < \varepsilon_c$, as was already corrected in ref. 28.]

We now turn to the supercritical regime, for which it is seen in Fig. 3 that the finite-size effect is not as strong as in the subcritical region, being consistent with the fact that the index s is much larger than that for $\varepsilon < \varepsilon_c$. In fact, the data are relatively stable against the change of N. This is why the correct value of γ' was obtained only for $\varepsilon > \varepsilon_c$ in ref. 16. σ is smaller than in the disordered phase because of entrainment. Figure 4b gives new evidence for (5.1).

Let us go on to the correlation function $C_Z(\tau)$. Results of simulations are presented in Fig. 5 for $N = 6000$ and for several values of ε below (Fig. 5a) and beyond (Fig. 5b) ε_c. Evidently the temporal coherence of fluc-

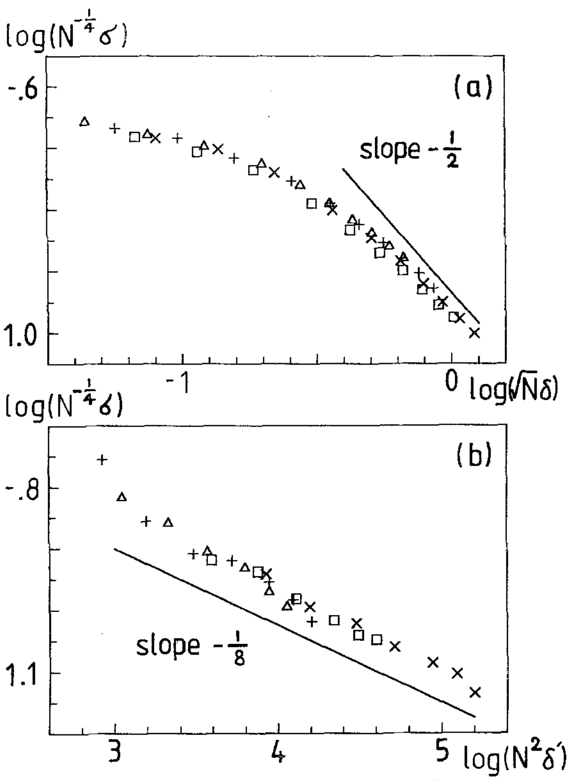

Fig. 4. Evidence for the finite-size scaling laws (a) (5.3) and (b) (5.1), where $\delta \equiv \varepsilon_c - \varepsilon$, $\delta' \equiv \varepsilon - \varepsilon_c$ and $\log \equiv \log_{10}$. (a) The subcritical regime: $N = 6000$ ($\triangle$), 10,000 (+), 14,000 ($\square$), 20,000 ($\times$). (b) The supercritical regime: $N = 1600$ ($\triangle$), 1900 (+), 3000 ($\square$), 6000 ($\times$). The straight lines show the theoretical slopes. σ was computed through averaging over nearly 130,000 ($\varepsilon < \varepsilon_c$) and 30,000–520,000 ($\varepsilon > \varepsilon_c$) iterations of (1.6).

tuations is enhanced as the threshold is approached. For $\varepsilon < \varepsilon_c$ the theory, (3.13), is also shown, which appears successful for not very large τ. For large τ the finite-size effect seems to be serious, which is no wonder, since if N is finite, fluctuations are actually quasiperiodic and their correlation should never vanish even for $\tau \to \infty$.

As a test of the scaling laws (3.18) and (4.62), we now investigate the critical behavior of the damping rate of correlation, denoted by Γ, which was obtained by least square fit as the slope of ln $|C_Z(\tau)|$ vs. $-\tau$. Since the finite-size effect was expected to be strong for $\varepsilon < \varepsilon_c$, the following plot was tried for the subcritical regime [recall (3.18) and (5.3)]:

$$\Gamma = N^{-1/2}\Psi_{dr}(N^{1/2}(\varepsilon_c - \varepsilon)) \tag{5.4}$$

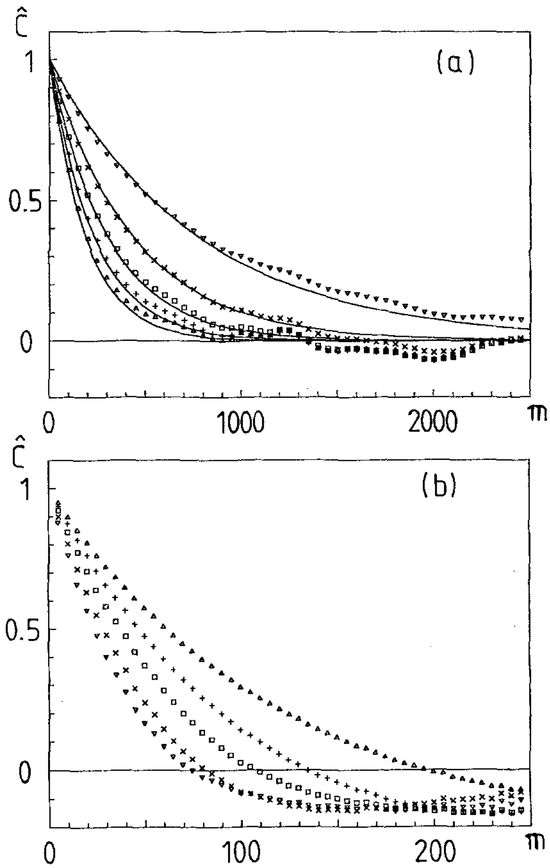

Fig. 5. Normalized correlation function of Z, $C(m) \equiv C_Z(m)/C_Z(0)$, where the argument $\tau \equiv m$ is measured in the unit of one iteration. In simulations N was fixed at 6000, and averaging was carried out over nearly 2^{15} iterations. (a) The subcritical regime: $\varepsilon = 0.002$ ($\triangle$), 0.004 ($+$), 0.006 ($\square$), 0.008 ($\times$), 0.01 (∇). The curves exhibit the theory, (3.13). (b) The supercritical regime: $\varepsilon = 0.014$ ($\triangle$), 0.016 ($+$), 0.018 ($\square$), 0.02 ($\times$), 0.022 (∇).

where the scaling function should satisfy $\Psi_{dr}(x) \sim x$ for $x \gg 1$. This law is verified in Fig. 6a for $N = 6000$–20000. As to the entrained phase, we simply show a log–log plot of Γ vs. ε for $N = 6000$ in Fig. 6b, where the expected behavior, $\Gamma \propto (\varepsilon - \varepsilon_c)^{1/4}$, is found. In this way it turns out that the theory is supported by simulations for $C_Z(\tau)$ as well.

So far we have addressed only σ and $C_Z(\tau)$. At the end of this section let us see what the distribution of $Z \equiv X + i'Y$ itself is like. It is possible to show on the basis of (3.5) and (4.21) that $\langle (X - \langle X \rangle)^2 \rangle = \langle (Y - \langle Y \rangle)^2 \rangle$ with $\langle (X - \langle X \rangle)(Y - \langle Y \rangle) \rangle = 0$ for general initial conditions. Therefore,

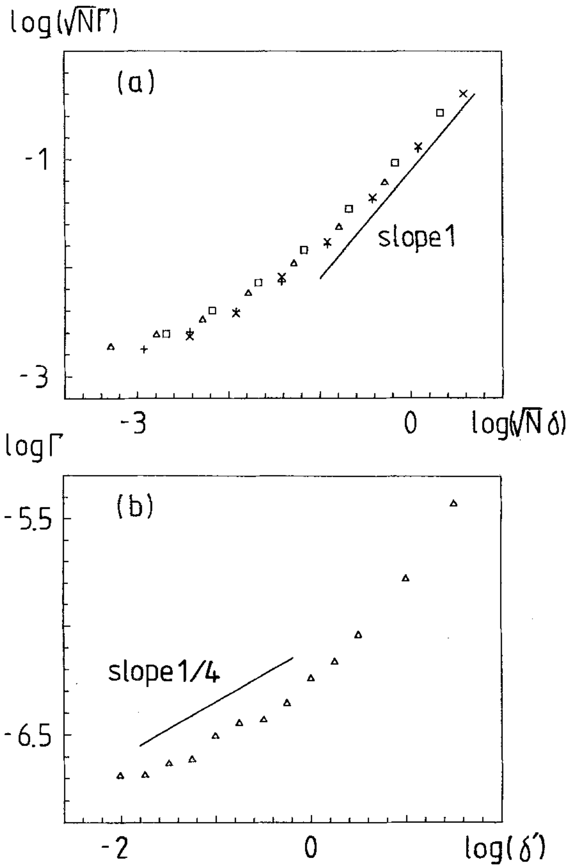

Fig. 6. Critical scalings of the damping rate of C_Z, Γ, computed from $C_Z(m)$ (averaged over 10^5–2×10^5 iterations) for (a) $m \leqslant 121$ and (b) $m \leqslant 31$. Here $\log \equiv \log_2$, $\delta \equiv \varepsilon_c - \varepsilon$, and $\delta' \equiv (\varepsilon - \varepsilon_c)/(\varepsilon^* - \varepsilon_c)$ with $\varepsilon^* = 0.015$. The straight lines show theoretical slopes. (a) The subcritical regime; evidence for the finite-size scaling, (5.4), where $N = 6000$ ($\triangle$), 10,000 ($+$), 14,000 ($\square$), 20,000 ($\times$). (b) The supercritical regime; $N = 6000$.

by way of the "central limit theorem," we may obtain as a distribution function of X and Y

$$P(X, Y) = \frac{N}{\pi\sigma^2} \exp\left\{-\frac{N}{\sigma^2}\left[(X - \langle X\rangle)^2 + (Y - \langle Y\rangle)^2\right]\right\} \tag{5.5}$$

where it should be noted that $\langle Z\rangle = \langle X\rangle + i'\langle Y\rangle = \hat{Z}$ depends on the initial condition through the phase factor. As a check of this normal law, the distribution of X is displayed in Fig. 7, where the numerical data excellently fit a Gaussian distribution function.

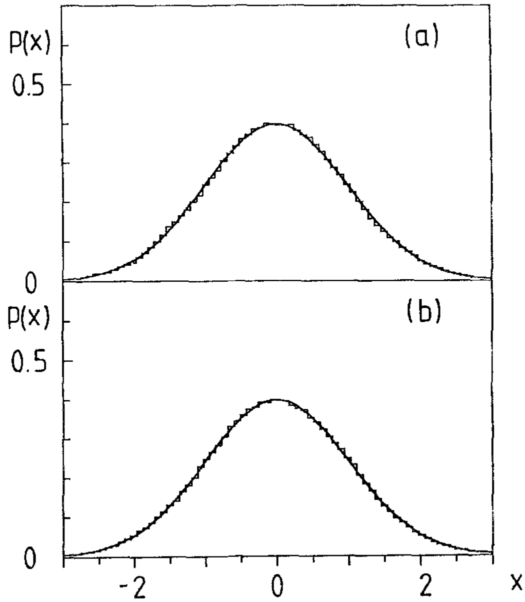

Fig. 7. Distribution of $x \equiv (X - \langle X \rangle)/\langle (X - \langle X \rangle)^2 \rangle^{1/2}$. The histograms show numerical outcomes obtained by averaging over 5×10^5 iterations for $N = 1000$. The curves exhibit the Gaussian law, $p(x) = \exp(-x^2/2)/(2\pi)^{1/2}$. (a) $\varepsilon = 0.01$ ($< \varepsilon_c$). (b) $\varepsilon = 0.015$ ($> \varepsilon_c$).

6. CORRELATION IN FREQUENCY SPACE

In this section we explore the origin of the unique feature of the phase transition, which will lead us to a new concept providing a physical interpretation of the finite-size scaling laws. Along this line it is clarified why the Kuramoto–Nishikawa theory contradicts our results.

6.1. Absence of Equality among Elements

Admittedly the most peculiar and curious aspect of the phase transition is that critical exponents related to fluctuations in the ordered phase deviate remarkably from the typical mean-field values taken in the disordered phase, in spite of the mean-field nature of the model equations. Ultimately it can be attributed to the fact that elements of our oscillator system are *not* identical to one another, due to the difference in their native frequencies. In other words, our system lacks "equality" among constituents in contrast to typical cooperative systems, say, ordinary spin systems. This fact is not important in the disordered phase, where every element simply oscillates, maintaining its own pace, and therefore equality is virtually established. In fact, we have already seen that critical exponents do not differ from those of classical spin systems with infinite-range interactions for $\varepsilon < \varepsilon_c$, which may stem from the virtual equality. On the other hand, in the ordered phase, inequalities among elements bring about a separation of the whole population into two classes: the entrained and the rest. This may remind us of macroscopic clusters in, say, an Ising system in which all the spins have a direction in common (upward or downward). In our system, however, the membership of each element, i.e., the class to which it belongs, is never renewed as time passes, unlike in such a spin system, being determined solely by whether $|\Delta_j| < Q$ or not [except for a relatively small number of oscillators near the boundary whose membership meanders because of fluctuations of $Z(t)$]. Thus, we have a decrease in the fluctuating degrees of freedom in the ordered phase, since the entrained oscillators exhibit only little fluctuations, which ultimately renders critical singularities weaker than in the ordered phase. In fact, it is possible to expand this argument analytically and elucidate the mechanism of suppressing singularities, as will be published elsewhere (for brevity).[32] In the next subsection we will reconsider the finite-size scalings from the viewpoint of the lack of equality.

6.2. Correlation Frequency and Finite-Size Scalings

It is convenient to reexpress (5.1) and (5.3) as follows:

$$\sigma = |\varepsilon - \varepsilon_c|^{-\gamma'} \tilde{\Psi}_{\pm}(N/N_c) \qquad (\varepsilon \gtrless \varepsilon_c) \tag{6.1}$$

with

$$N_c = |\varepsilon - \varepsilon_c|^{-1/s} \tag{6.2}$$

where the scaling functions satisfy $\tilde{\Psi}_{\pm}(x) \propto x^{s\gamma'}$ $(x \ll 1)$ as well as $\tilde{\Psi}_{\pm}(\infty) < \infty$. These forms of finite-size scaling may be particularly useful in discussing how fast σ converges for $N \to \infty$. The key is of course N_c: what is its physical meaning? For instance, the same form of scaling is known to hold for spin systems with infinite-range interactions,[33] but no clear-cut physical interpretation of N_c has been found. This is because the concept of "distance" is missing in such mean-field models. Concerning our system, however, the characteristic emphasized above enables us to arrive at a natural interpretation of N_c despite the fact that interactions are of a mean-field type.

Imagine a one-dimensional array of oscillators such that their positions are just specified by $\Delta_j = \Omega_j - \tilde{\Omega}$. Then the closer together two of the oscillators are on the array, the stronger their correlation should be, because a small difference in their frequencies means that they tend to be entrained or deentrained at the same time in the course of the variation of the mean-field $Z(t)$. Therefore, we may introduce a correlation length in frequency space. A problem is that the frequency space lacks translational invariance, which makes the definition of the correlation length ambiguous. Taking the form of $f(\Omega)$ assumed into account, however, it would be most natural and to the point to consider how far correlation propagates from the central oscillator with $\Delta = 0$. Suppose that it may be regarded as extending up to $\Delta = \pm\Delta_c$ on the array. We then call Δ_c a "correlation frequency." It is now clear that as N is varied, finite-size effects begin to work when

$$N \sim N \int_{-\Delta_c}^{\Delta_c} d\Delta\, \tilde{f}(\Delta)$$

In other words, we may put

$$N_c = N \int_{-\Delta_c}^{\Delta_c} d\Delta\, \tilde{f}(\Delta) \tag{6.3}$$

Let us investigate the critical behavior of the correlation frequency, from which the crossover exponent s comes out by way of (6.3). For this purpose it should be noted that if we put

$$Z(t) - \langle Z \rangle \equiv (\sigma/\sqrt{N})\, z(t/\tau_c) \tag{6.4}$$

$z(x)$ may be regarded as a universal Gaussian random process. For convenience, suppose that $j = 0$ is the central oscillator with $\Delta = 0$. Then we obtain, using (6.4) and (2.1),

$$\frac{d\tilde{\theta}_0}{du} = \mathrm{Im}\left\{\left[\frac{\varepsilon\tau_c}{2\pi}\langle Z\rangle + \frac{\varepsilon\sigma\tau_c}{2\pi\sqrt{N}}\, z(u)\right] e^{-2\pi i'\tilde{\theta}_0}\right\} \tag{6.5}$$

and

$$\frac{d\tilde{\theta}_j}{du} = \tau_c\Delta_j + \mathrm{Im}\left\{\left[\frac{\varepsilon\tau_c}{2\pi}\langle Z\rangle + \frac{\varepsilon\sigma\tau_c}{2\pi\sqrt{N}}\, z(u)\right] e^{-2\pi i'\tilde{\theta}_j}\right\} \tag{6.6}$$

where $u \equiv t/\tau_c$. Obviously, it is the common term with $z(u)$ that produces correlation between the two oscillators. From these equations it follows that any type of equal-time correlation function, e.g.,

$$\begin{aligned} C_f(\Delta_j) \equiv{} & \langle (e^{2\pi i'\tilde{\theta}_j} - \langle e^{2\pi i'\tilde{\theta}_j}\rangle)(e^{2\pi i'\tilde{\theta}_0} - \langle e^{2\pi i'\tilde{\theta}_0}\rangle)^* \rangle \\ & \times \langle |e^{2\pi i'\tilde{\theta}_0} - \langle e^{2\pi i'\tilde{\theta}_0}\rangle|^2\rangle^{-1} \end{aligned} \tag{6.7}$$

depends only on three parameters: $\tau_c \Delta_j$, $\varepsilon\langle Z\rangle\tau_c$, and $\varepsilon\sigma\tau_c/\sqrt{N}$. Moreover, since Δ_c vanishes in the limit $N \to \infty$, we have from (6.3)

$$N_c \cong 2N\bar{f}(0)\,\Delta_c \tag{6.8}$$

which means $\Delta_c \propto N^{-1}$. Consequently, Δ_c has to be of the form

$$\tau_c \Delta_c \cong (\varepsilon\sigma\tau_c/\sqrt{N})^2\, g(\varepsilon\langle Z\rangle\tau_c)$$

with a function $g(x)$ such that $0 < g(0) < \infty$, which may lead to

$$\Delta_c \cong \sigma^2\tau_c/N \tag{6.9}$$

near ε_c (apart from a constant factor), irrespective of the sign of $\varepsilon - \varepsilon_c$. Combining (6.9) with (6.8) and defining critical indices p and ν by

$$\tau_c \sim |\varepsilon - \varepsilon_c|^{-p}, \qquad N\Delta_c \sim |\varepsilon - \varepsilon_c|^{-\nu} \tag{6.10}$$

we find via (6.2)

$$\nu = 1/s = 2\gamma' + p \qquad (\varepsilon \gtrless \varepsilon_c) \tag{6.11}$$

which is easily confirmed to reproduce the values of s presented in Section 5. The above result may be viewed as a kind of "scaling law relation."[25]

Figure 8 shows some examples of C_f defined in (6.7) (note that the abscissa is not Δ, but j), where it is seen that correlation with the central

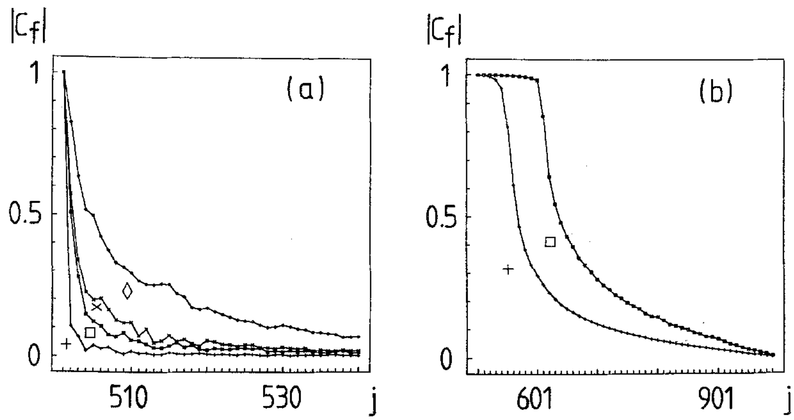

Fig. 8. Correlation between the central oscillator and others [$|C_f(\Delta_j)|$ is shown: see (6.7)]. All the data were obtained for $N = 1000$ and by averaging over 192×10^4–480×10^4 iterations. The oscillator of $j = 501$ was used as the central one. (a) The subcritical regime; $\varepsilon = 0.002$ (+), 0.008 (□), 0.01 (×), 0.012 (◇). (b) The supercritical regime; $\varepsilon = 0.0127$ (+), 0.013 (□). (For larger ε the convergence in computations was so slow that the results are not displayed.)

oscillator becomes more and more far-reaching as the threshold of mutual entrainment is approached. This may qualitatively support the critical behavior of Δ_c with $\nu > 0$ discussed above, but we have not yet succeeded in verifying it on a quantitative level, because of very slow convergence of averages in computing C_f. In Fig. 8b the flat regimes are created by mutual entrainment. Further investigations will be made elsewhere on the correlation in frequency space.

6.3. On the Kuramoto–Nishikawa Theory

As mentioned earlier, Kuramoto and Nishikawa (KN)[27] published a statistical mechanical theory for the order parameter after Daido's simulations.[15] Among other things, their theory predicts the absence of critical divergence for σ. In what follows we aim at clarifying why the KN theory contradicts our results as well as Daido's conjecture based on his own simulations.[15,16]

KN divide the instantaneous order parameter as

$$Z(t) = Z_s(t) + Z_d(t) \tag{6.12}$$

where Z_s and Z_d are contributions by entrained and nonentrained oscillators, respectively. It is the latter that dominates the critical fluctuations of Z, so that we need to check the way KN deal with Z_d. In Section 5 of their paper, where the fluctuations are discussed in most detail, KN invoke the following approximate invariant measure for the set of nonentrained oscillators:

$$\rho_{do}(\tilde{\theta}_{N_s+1},..., \tilde{\theta}_N) \equiv \prod_{j=N_s+1}^{N} \frac{\{\Delta_j^2 - (\varepsilon |\langle Z \rangle| / 2\pi)^2\}^{1/2}}{|v_j(\tilde{\theta}_j)|} \tag{6.13}$$

where N_s is the number of entrained oscillators, and oscillators with $j = N_s + 1,..., N$ are supposed to be nonentrained ones for convenience, and

$$v_j(\tilde{\theta}_j) \equiv \Delta_j + (\varepsilon/2\pi) \operatorname{Im}\{\langle Z \rangle e^{-2\pi i' \tilde{\theta}_j}\} \tag{6.14}$$

which is the rhs of Eq. (2.1) with $Z(t)$ replaced by $\langle Z \rangle$. Adopting the measure is equivalent to assuming that $d\tilde{\theta}_j/dt = v_j(\tilde{\theta}_j)$ for $j = N_s + 1,..., N$. On the basis of such an approximation KN show essentially

$$\langle |Z - \langle Z \rangle|^2 \rangle = (1 - R)/N \tag{6.15}$$

or

$$\sigma = (1 - R)^{1/2} \tag{6.16}$$

with $R \equiv \lim_{N \to \infty} N_s/N$, which implies that σ is finite even at $\varepsilon = \varepsilon_c$. Note, however, that the approximate measure does not carry the effect of correlation among oscillators at all, since it is completely decoupled with respect to the variables $\tilde{\theta}_{Ns+1},..., \tilde{\theta}_N$. [In fact, it is straightforward to derive (6.15), provided we are allowed to neglect the correlation among elements.] As we have seen in the previous subsection, such a correlation actually does play a crucial role at the onset of mutual entrainment, just as in ordinary phase transitions. Therefore, it is no wonder that the KN theory does not accord with our results. Furthermore, it has been reported recently by Nishikawa that their theory also contradicts simulations on the macroscopic relaxation of $Z(t)$.(34) In view of these facts, a new attempt has been sketched out very recently by KN.(35)

7. SUMMARY, DISCUSSION, AND FURTHER PROBLEMS

In summary, a statistical mechanical theory has been developed for intrinsic (spontaneous) fluctuations of an order parameter in large populations of limit-cycle oscillators with uniform infinite-range interactions and with distributed native frequencies as modeled by Eq. (1.2). On this basis an extensive study has been made on the behavior of fluctuations at the onset of mutual entrainment. For simplicity it has been premised that the distribution of native frequencies is symmetric with a parabolic peak point as well as monotonic on both sides of the latter. Although this premise seems to be met by typical distributions of scientific relevance, for completeness it will be studied elsewhere what may happen if part of the premise is loosened.

Let us now discuss the significance of the present work in detail.

7.1. Methodology: System-Size Expansion

The theory is based on a kind of system-size expansion, which is a powerful tool especially because it turns large degrees of freedom, which is apparently a source of difficulty of analytic studies, into the reason why such studies are actually possible. Our method is clearly different from van Kampen's,(30) since the former is for deterministic dynamical systems, while the latter treats kinetic equations describing stochastic processes such as master equations and Fokker–Planck equations. No doubt what has made our approach successful is the mean-field structure of the model equation. On second thought, we may expect that our method will be widely applicable to mean-field type models in a variety of fields beyond a narrow scope of population dynamics of coupled oscillators. It may be an important key to the approach to dynamical systems with very large degrees of freedom.

7.2. Theory of Intrinsic Fluctuations

Our theory may be significant at least in two respects. First, it may greatly help in the analysis of data obtained by computer simulations or laboratory experiments. It would also be useful for controlling a system, since it enables us to foresee the occurrence of a phase transition by checking the behavior of fluctuations. Second, it has extended the analogy between the onset of mutual entrainment and ordinary phase transitions up to the level of fluctuations by revealing critical power-law divergence of fluctuations at the former. Before this work the analogy has been known to hold only for the behavior of the order parameter. Therefore, we may say that by this work the onset of mutual entrainment has been given a more respectable qualification as a new type of phase transition that deserves extensive investigation.

7.3. Discrepancy between Subcritical and Supercritical Exponents

It has been found, however, that the "phase transition" possesses a curious feature. Namely, critical exponents related to fluctuations deviate remarkably from mean-field values in the ordered phase, though they equal the latter in the disordered phase (see Table 1). This forms a striking contrast to thermal phase transitions, whose origin consists in the lack of equality among elements, as pointed out in Section 6. What may be a possible significance of the unique feature, for instance, in a biological context? The fact that critical divergence is much weaker in the ordered phase may suggest that the rhythmic order achieved therein is fairly robust and stable except for a very close neighborhood of the critical point. This

Table I. Critical Exponents[a]

Exponent	$\varepsilon < \varepsilon_c$	$\varepsilon > \varepsilon_c$	Definition
γ'	1/2	1/8	$\sigma \propto \lvert\varepsilon - \varepsilon_c\rvert^{-\gamma'}$
p	1	1/4	$\tau_c \propto \lvert\varepsilon - \varepsilon_c\rvert^{-p}$
s	1/2	2	$\lvert\varepsilon_c(f) - \varepsilon_c\rvert \propto N^{-s}$
$\nu = 1/s$	2	1/2	$N_c \propto \lvert\varepsilon - \varepsilon_c\rvert^{-\nu}$, $N\Delta_c \propto N^{-1} \lvert\varepsilon - \varepsilon_c\rvert^{-\nu}$

[a] $\varepsilon_c(f)$ is the boundary value at which the finite-size effect begins to come into play.

seems quite favorable to living organisms, whose temporal coherence needs to be maintained to great accuracy.

7.4. Correlation Frequency

Our oscillator system may be somewhat weird in the sense that it has a sort of correlation length despite its mean-field nature. It should be emphasized that this feature is not a mere addition, but is deeply connected with the existence of the phase transition. In fact, the concept of correlation frequency stems from the lack of equality among elements due to the distribution of native frequency, without which there is no phase transition at all (i.e., $\varepsilon_c = 0$). This is why Δ_c plays a crucial role in the critical scalings. As mentioned earlier, however, no translational invariance exists in frequency space, so that what Δ_c means is slightly different from ordinary correlation lengths. In fact, we could define Δ_c in a different way. Further investigations will be devoted to clarifying the nature of correlation in frequency space.

A number of problems remain to be addressed, of which the following three are particularly important.

(a) To examine to what extent the approximation (4.9) is efficient in a *global* range of the control parameter in the ordered phase, since we have only checked its validity near ε_c because of the difficulty in comparing the complicated formulas (4.31) and (4.60) with the results of simulations.

(b) To see whether the critical behaviors are modified or not for finite-range interactions.

(c) To elucidate a relationship between the fluctuations of Z and its macroscopic response to external perturbations.

The second subject may be important to see how robust our results are, and the third one is proposed to establish the physical significance of the fluctuations by searching for an analogue of the fluctuation-dissipation theorem.[36] Future publications will be devoted to these subjects.

APPENDIX

A1. Definition of $Y_m(\Delta)$ and Its Properties

By (2.11), $\tilde{\psi}_j \equiv \psi_j - \alpha$ obeys

$$d\tilde{\psi}_j/dt = \Delta_j - Q \sin 2\pi\tilde{\psi}_j \qquad (|\Delta_j| > Q) \tag{A.1}$$

Let the solution to this equation satisfying $\tilde{\psi}(0)=p$ be $\Theta(t, \Delta_j, p)$, and we can easily find

$$\Theta(t, -\Delta, -p) = -\Theta(t, \Delta, p) \tag{A.2}$$

It follows from this relationship that if we put

$$e^{2\pi i' \Theta(t, \Delta, 0)} \equiv \sum_{m=-\infty}^{\infty} Y_m(\Delta) \exp[2\pi i' m(\Delta^2 - Q^2)^{1/2} t] \tag{A.3}$$

the coefficients have to obey

$$Y^*_{-m}(\Delta) = Y_m(-\Delta) \tag{A.4}$$

Now suppose that after starting at p for $t=0$, $\Theta(t, \Delta, p)$ first arrives at an integer value $\equiv l(p, \Delta)$ when t is $\tau(p, \Delta)$. This allows us to put

$$\Theta(t, \Delta, p) = \Theta(t - \tau(p, \Delta), \Delta, 0) + l(p, \Delta) \tag{A.5}$$

Then, noting $\tilde{\psi}_j(t) = \Theta(t, \Delta_j, p_j)$ with $p_j = \tilde{\psi}_j(0)$, we have

$$d_m(\Delta_j) = \{\exp[-2\pi i' m(\Delta_j^2 - Q^2)^{1/2}\, \tau(p_j, \Delta_j)] \exp(2\pi i' \alpha)\}\; Y_m(\Delta_j) \tag{A.6}$$

by (4.24) as well as (A.3) and (A.5).

A2. Derivation of (4.34)

By the definition of Θ in Appendix A1 we may write

$$\Theta(t, \Delta, 0) = \hat{\Theta}(Qt, \Delta/Q) \tag{A.7}$$

where $\hat{\Theta}(\tau, x)$ is a universal function defined as the solution to

$$d\psi/d\tau = x - \sin 2\pi\psi, \qquad \psi(\tau = 0) = 0 \tag{A.8}$$

for $x > 1$. Then, by (A.3), we have

$$\begin{aligned} Y_m(\Delta) &= (\Delta^2 - Q^2)^{1/2} \int_0^{1/(\Delta^2 - Q^2)^{1/2}} dt \\ &\qquad \times \exp[2\pi i' \Theta(t, \Delta, 0) - 2\pi i' m(\Delta^2 - Q^2)^{1/2} t] \\ &= [(\Delta/Q)^2 - 1]^{1/2} \int_0^{1/[(\Delta/Q)^2 - 1]^{1/2}} \\ &\qquad \times d\tau \exp\{2\pi i' \hat{\Theta}(\tau, \Delta/Q) - 2\pi i' m[(\Delta/Q)^2 - 1]^{1/2} \tau\} \\ &\equiv \hat{Y}_m(\Delta/Q) \end{aligned} \tag{A.9}$$

where $\hat{Y}_m(x)$ clearly is a universal function of x and m.

A3. Derivation of (4.39)

By (A.9) we have for $y > 0$

$$\hat{Y}_m((1+y^2)^{1/2}) = y\int_0^{1/y} d\tau \exp[2\pi i'\hat{\Theta}(\tau, (1+y^2)^{1/2}) - 2\pi i' m y\tau] \qquad \text{(A.10)}$$

For convenience we introduce $u(\tau)$ by

$$u(\tau) = \hat{\Theta}(\tau+\tau_0, (1+y^2)^{1/2}) - \tfrac{1}{4} \qquad \text{(A.11)}$$

to obtain

$$\hat{Y}_m((1+y^2)^{1/2}) = i' y e^{-2\pi i' m y \tau_0}\int_0^{1/y} d\tau\, e^{2\pi i' u(\tau) - 2\pi i' m y\tau} \qquad \text{(A.12)}$$

where τ_0 in (A.11) is chosen so as to assure $0 < -u(0) \equiv c \ll 1$, and note that by (A.8), u obeys

$$du/d\tau = (1+y^2)^{1/2} - \cos 2\pi u \qquad \text{(A.13)}$$

From now on we consider the dynamics of u in the intervals $0 \leqslant \tau \leqslant \tau_1 \equiv u^{-1}(c)$ and $\tau_1 \leqslant \tau \leqslant y^{-1}$ to evaluate the integral in (A.12) in the limit $y \to +0$. In the former interval, Eq. (A.13) may be approximated by

$$du/d\tau = \tfrac{1}{2} y^2 + 2\pi^2 u^2 \qquad \text{(A.14)}$$

so that we obtain

$$u(\tau) = \frac{y}{2\pi}\tan\left\{\pi y\tau - \tan^{-1}\left(\frac{2\pi c}{y}\right)\right\} \qquad (0 \leqslant \tau \leqslant \tau_1) \qquad \text{(A.15)}$$

with

$$\tau_1 = \frac{2}{\pi y}\tan^{-1}\left(\frac{2\pi c}{y}\right) \qquad \text{(A.16)}$$

With the aid of (A.15) and (A.16), we find

$$\begin{aligned} & y\int_0^{\tau_1} d\tau\, e^{2\pi i' u(\tau) - 2\pi i' m y\tau} \\ &= \frac{2}{\pi} e^{-2mi'\tan^{-1}(2\pi c/y)}\int_0^{\tan^{-1}(2\pi c/y)} dx \cos(y\tan x - 2mx) \\ &= \frac{2}{\pi}\left\{-\frac{1}{2\pi c} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2\pi c)^{2n+1}}{(2n+2)!\,(2n+1)}\right. \\ &\quad \left. + (-1)^m \int_0^{\pi/2} dx \sin(2mx)\tan x\right\} y + o(y) \end{aligned} \qquad \text{(A.17)}$$

The remaining part of the integral may be evaluated as follows:

$$
\begin{aligned}
y\int_{\tau_1}^{1/y} d\tau\, e^{2\pi i' u(\tau)-2\pi i' m y\tau}
&= y e^{-2\pi i' m y \tau_1}\int_0^{1/y-\tau_1} d\tau\, e^{2\pi i' u(\tau+\tau_1)-2\pi i' m y \tau}\\
&= y\int_0^{1/\pi^2 c} d\tau\, e^{2\pi i' \tilde{u}(\tau)} + o(y)
\end{aligned}
\tag{A.18}
$$

where (A.16) has been used and $\tilde{u}(\tau)$ is defined by

$$
d\tilde{u}/d\tau = 1-\cos 2\pi\tilde{u}, \qquad \tilde{u}(0)=c \tag{A.19}
$$

From Eqs. (A.12), (A.17), and (A.18) we finally obtain

$$
\hat{Y}_m((1+y^2)^{1/2}) = i'\left\{\mathscr{D} + (-1)^m \frac{2}{\pi}\int_0^{\pi/2} dx\, \sin(2mx)\tan x\right\} y + o(y) \tag{A.20}
$$

where $\mathscr{D}$ is a constant which does not depend on m.

A4. Derivation of (4.40) and (4.41)

Let us first treat $\tilde{C}(2\pi Qz)$. By (4.27)

$$
\begin{aligned}
\tilde{C}_{\mathrm{e}}(2\pi Qz) &= \frac{1}{2\pi}\int_{|x|<1} dx\, \frac{\tilde{J}(Qx)}{(1-x^2)^{1/2}+i'z}\\
&= \frac{1}{2\pi}\int_{|x|<1} dx\, \frac{\tilde{J}(Qx)}{(1-x^2)^{1/2}} + i'\frac{\tilde{J}(Q)}{\pi} z\ln|z|\\
&\quad + \left\{-\frac{1}{2}\tilde{J}(Q)\operatorname{sgn}(z) - i'\sum_{k=1}^{\infty}\frac{\alpha_k}{k}\right\} z + o(z)
\end{aligned}
\tag{A.21}
$$

where the α_k are defined through

$$
\tilde{J}(Q(1-y^2)^{1/2})/(1-y^2)^{1/2} = \tilde{J}(Q) + 2\pi\sum_{k=1}^{\infty}\alpha_k y^{2k} \tag{A.22}
$$

By (4.28)

$$
\begin{aligned}
\tilde{C}_{\mathrm{ne}}(2\pi Qz) &= \frac{1}{2}\frac{|z|}{(1+z^2)^{1/2}}\tilde{J}(Q(1+z^2)^{1/2})\\
&\quad + \frac{i'}{2\pi}\,\mathrm{PV}\int_{-\infty}^{\infty} dy\, \frac{|y|}{(1+y^2)^{1/2}}\tilde{J}(Q(1+y^2)^{1/2})\frac{1}{y-z}
\end{aligned}
\tag{A.23}
$$

The second term of the above expression may be cast into the form

$$\frac{i'}{2\pi}\int_0^\infty dy\,\{J(y,z)-J(y,-z)\}\frac{1}{y}\equiv i'\Phi(z) \tag{A.24}$$

where

$$J(y,z)\equiv\frac{|y+Qz|}{[(y+Qz)^2+Q^2]^{1/2}}\tilde{J}([(y+Qz)^2+Q^2]^{1/2})$$

For the present let us assume $z>0$, and we have

$$\Phi(z)=\tilde{\Phi}_1(z)+\tilde{\Phi}_2(z) \tag{A.25}$$

where

$$\begin{aligned}\tilde{\Phi}_1(z)&\equiv\frac{1}{2\pi}\int_0^{Qz} dy\,\{J(y,z)-J(y,-z)\}\frac{1}{y}\\ &=\frac{1}{\pi}\tilde{J}(Q)z+o(z)\end{aligned} \tag{A.26}$$

and

$$\begin{aligned}\tilde{\Phi}_2(z)&\equiv\frac{1}{2\pi}\int_{Qz}^\infty dy\,\{J(y,z)-J(y,-z)\}\frac{1}{y}\\ &=-\frac{\tilde{J}(Q)}{\pi}z\ln z+\Gamma z+o(z)\end{aligned} \tag{A.27}$$

with

$$\begin{aligned}\tilde{\Gamma}\equiv&-\frac{\tilde{J}(Q)}{\pi}\ln Q-\frac{Q^3}{\pi}\int_0^\infty dy\,(\ln y)\frac{d}{dy}\left\{\frac{\tilde{J}((y^2+Q^2)^{1/2})}{(y^2+Q^2)^{3/2}}\right\}\\ &+\frac{Q}{\pi}\int_0^\infty dy\frac{y}{y^2+Q^2}\tilde{J}'((y^2+Q^2)^{1/2})\end{aligned} \tag{A.28}$$

For $z>0$ we may use the relation $\Phi(z)=-\Phi(-z)$, as is clear by (A.24). Finally, we find

$$i'\Phi(z)=-i'\frac{\tilde{J}(Q)}{\pi}z\ln|z|+i'\left\{\frac{\tilde{J}(Q)}{\pi}+\tilde{\Gamma}\right\}z+o(z)$$

which leads to

$$\tilde{C}_{\mathrm{ne}}(2\pi Qz)=-i'\frac{\tilde{J}(Q)}{\pi}z\ln|z|+\left\{\frac{1}{2}\tilde{J}(Q)\operatorname{sgn}(z)+i'\left[\frac{\tilde{J}(Q)}{\pi}+\tilde{\Gamma}\right]\right\}z+o(z) \tag{A.29}$$

In this way we obtain on the basis of (A.21) and (A.29)

$$\tilde{C}(2\pi Qz) = \frac{1}{2\pi}\int_{|x|<1} dx \frac{\tilde{J}(Qx)}{(1-x^2)^{1/2}} + i'\left\{\frac{\tilde{J}(Q)}{\pi} + \tilde{\Gamma} - \sum_{k=1}^{\infty}\frac{\alpha_k}{k}\right\} z + o(z) \tag{A.30}$$

It is not very difficult to show using (A.22) and (A.28) that the coefficient of z in (A.30) vanishes for $\varepsilon = \varepsilon_c$. We have thus derived (4.40). Equation (4.41) can be readily obtained as follows: by (4.29)

$$\begin{aligned}\tilde{U}(2\pi Qz) &= \frac{1}{2\pi}\int_{|x|<1} dx \frac{\tilde{J}(Qx)(1-2x^2)}{(1-x^2)^{1/2} + i'z} \\ &= \frac{1}{2\pi}\int_{|x|<1} dx \frac{\tilde{J}(Qx)(1-2x^2)}{(1-x^2)^{1/2}} - i'\frac{\tilde{J}(Q)}{\pi} z \ln|z| \\ &\quad + \left\{\frac{1}{2}\tilde{J}(Q)\,\mathrm{sgn}(z) - i'\sum_{k=1}^{\infty}\frac{\alpha'_k}{k}\right\} z + o(z)\end{aligned} \tag{A.31}$$

where α'_k are defined by

$$\frac{\tilde{J}(Q(1-y^2)^{1/2})(-1+2y^2)}{(1-y^2)^{1/2}} = -\tilde{J}(Q) + 2\pi\sum_{k=1}^{\infty}\alpha'_k y^{2k}$$

which leads to (4.41).

A5. Derivation of (4.46)

By (A.9)

$$\hat{Y}_m(x) = (x^2-1)^{1/2}\int_0^{1/(x^2-1)^{1/2}} d\tau \exp[2\pi i'\hat{\Theta}(\tau, x) - 2\pi i' m(x^2-1)^{1/2}\tau] \tag{A.32}$$

where $\hat{\Theta}(\tau, x)$ is the solution to Eq. (A.8). Let us derive the asymptotic form of $\hat{Y}_m(x)$ for $x \gg 1$. For this purpose we may approximate Θ as

$$\hat{\Theta}(\tau, x) \cong x\tau \tag{A.33}$$

obtaining from (A.32)

$$\hat{Y}_m(x) \cong \frac{1}{2(1-m)x^2+1} \tag{A.34}$$

which results in (4.46).

A6. Derivation of (4.53) and (4.54)

By (4.27) and (4.28) we have

$$\tilde{C}_{e}(2\pi\sqrt{Q}\,z)=\frac{\tilde{J}(0)}{\pi i' z}\sqrt{Q}+o(\sqrt{Q}) \tag{A.35}$$

and

$$\begin{aligned}\tilde{C}_{ne}(2\pi\sqrt{Q}\,z)=&\frac{1}{2(1+Qz^{-2})^{1/2}}\tilde{J}([Q(z^2+Q)]^{1/2})\\&+\frac{i'}{2\pi}\mathrm{PV}\int_{-\infty}^{\infty}dy\,\frac{|y|}{(1+y^2)^{1/2}}\tilde{J}(Q(1+y^2)^{1/2})\frac{1}{y-(z/\sqrt{Q})}\end{aligned} \tag{A.36}$$

The second term of (A.36) (without i') may be expressed as [see (A.24)]

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{\infty}dy\left\{\frac{|y+\sqrt{Q}\,z|}{[(y+\sqrt{Q}\,z)^2+Q^2]^{1/2}}\tilde{J}([(y+\sqrt{Q}\,z)^2+Q^2]^{1/2})\right.\\&\qquad\left.-\frac{|y-\sqrt{Q}\,z|}{[(y-\sqrt{Q}\,z)^2+Q^2]^{1/2}}\tilde{J}([(y-\sqrt{Q}\,z)^2+Q^2]^{1/2})\right\}\frac{1}{y}\\&\equiv\int_0^{\infty}dy\,J_1(y,z)\end{aligned} \tag{A.37}$$

Suppose $z>0$ for the time being. Then we can show

$$\int_0^{\sqrt{Q}\,z}dy\,J_1(y,z)=\frac{\sqrt{Q}}{2\pi z}\tilde{J}(0)+o(\sqrt{Q}) \tag{A.38}$$

and

$$\int_{\sqrt{Q}\,z}^{\infty}dy\,J_1(y,z)=\frac{\sqrt{Q}\,z}{\pi}\int_0^{\infty}dy\frac{\tilde{J}'(y)}{y}+o(\sqrt{Q}) \tag{A.39}$$

Using (A.37)–(A.39) and noting $J_1(y,-z)=-J_1(y,z)$, we obtain an expression for the second term in (A.36) which results in

$$\tilde{C}_{ne}(2\pi\sqrt{Q}\,z)=\frac{1}{2}\tilde{J}(0)+i'\frac{\sqrt{Q}}{\pi}\left\{\frac{\tilde{J}(0)}{2z}+z\int_0^{\infty}dy\frac{\tilde{J}'(y)}{y}\right\}+o(\sqrt{Q}) \tag{A.40}$$

irrespective of the sign of z. By (A.35) and (A.40) we arrive at (4.53). In the same way we can obtain (4.54) from (4.29):

$$\tilde{U}(2\pi\sqrt{Q}\,z)=\frac{\tilde{J}(0)}{3\pi i' z}\sqrt{Q}+o(\sqrt{Q}) \tag{A.41}$$

APPENDIX B. REFINING THE SYSTEM-SIZE EXPANSION

The purpose of this final appendix is twofold:

1. We propose a refined version of the system-size expansion developed in Section 2.2, which is different from the original one in the respect that the secular terms of the form t/N^q $(q>0)$ are completely suppressed in every order of the expansion.

2. On this basis, we show that the results for σ, C_Z, etc., in the text remain valid.

Let us start by putting

$$Z(t)=\hat{Z}+\frac{1}{\sqrt{N}}w_1+\frac{1}{N}w_2+\cdots \tag{B.1}$$

$$\tilde{\theta}_j(t)=\psi_j+\frac{1}{\sqrt{N}}\phi_j^{(1)}+\frac{1}{N}\phi_j^{(2)}+\cdots \tag{B.2}$$

The key point is how to define the dominant phase $\psi_j(t)$. Instead of the naive way adopted in Section 2.2, we here define it as follows:

$$d\psi_j/dt=\Delta_j'+(\varepsilon/2\pi)\,\mathrm{Im}\{\hat{Z}e^{-2\pi i'\psi_j}\} \tag{B.3}$$

with

$$\Delta_j'\equiv\Delta_j+\frac{1}{\sqrt{N}}\Delta_j^{(1)}+\frac{1}{N}\Delta_j^{(2)}+\cdots \tag{B.4}$$

where for $k=1, 2,\ldots$ we choose the coefficient $\Delta_j^{(k)}$ in such a way that $\phi_j^{(k)}$ in (B.2) can evade secular terms. Substituting (B.1) and (B.2) into (2.1) and making an expansion in N, we obtain

$$d\phi_j^{(1)}/dt=-\Delta_j^{(1)}+(\varepsilon/2\pi)\,\mathrm{Im}\{(w_1-2\pi i'\hat{Z}\phi_j^{(1)})e^{-2\pi i'\psi_j}\} \tag{B.5}$$

and so on. Hereafter we are concerned only with $\phi_j^{(1)}$, $\Delta_j^{(1)}$, w_1 for simplicity. The higher order terms may be treated in the same way. Solving (B.5) leads to

$$\phi_j^{(1)}=\int_0^t dt'\left[\exp\int_{t'}^t a_j(\tau)\,d\tau\right]b_j(t')-\Delta_j^{(1)}\int_0^t dt'\exp\int_{t'}^t a_j(\tau)\,d\tau \tag{B.6}$$

where

$$\begin{aligned} a_j(t)&\equiv-\varepsilon\,\mathrm{Re}\{\hat{Z}e^{-2\pi i'\psi_j(t)}\}\\ b_j(t)&\equiv(\varepsilon/2\pi)\,\mathrm{Im}\{w_1(t)e^{-2\pi i'\psi_j(t)}\}\end{aligned} \tag{B.7}$$

In order to fix $\Delta_j^{(1)}$, we consider the two regimes $\varepsilon<\varepsilon_c$ and $\varepsilon>\varepsilon_c$ separately.

B1. The Subcritical Regime ($\epsilon < \epsilon_c$)

Since $\hat{Z} = 0$ in this regime, (B.6) reduces to

$$\phi_j^{(1)} = \int_0^t dt'\, b_j(t') - \Delta_j^{(1)} t \tag{B.8}$$

so that we have to choose $\Delta_j^{(1)}$ as

$$\Delta_j^{(1)} = (\varepsilon/2\pi)\,\mathrm{Im} < w_1 e^{-2\pi i'\psi_j} \rangle \tag{B.9}$$

where it should be recalled that the brackets $\langle \cdots \rangle$ stand for a long-time average.

B2. The Supercritical Regime ($\epsilon > \epsilon_c$)

In this regime the whole population may be divided into the group of entrained oscillators and that of nonentrained ones. For the former, i.e., oscillators with $|\Delta_j'| < Q$, we find by substituting the solution to (B.3) into (B.7) that

$$a_j(t) = -2\pi(Q^2 - \Delta_j'^2)^{1/2} \equiv -\Lambda_j = \text{const} \tag{B.10}$$

so that the first term on the rhs of (B6) may be expressed as

$$\int_0^t dt'\, e^{-\Lambda_j t'}\, b_j(t - t')$$

which does not produce a secular term as $t \to \infty$ because $\Lambda_j > 0$. Therefore, we may put $\Delta_j^{(1)} = 0$ for entrained oscillators. For the latter consisting of oscillators with $|\Delta_j'| > Q$ let us first note that the rhs of (B6) may read

$$\left[\exp \int_0^t a_j(\tau)\, d\tau\right] \left\{ \int_0^t dt' \left[\exp - \int_0^{t'} a_j(\tau)\, d\tau\right] b_j(t') \right.$$
$$\left. - \Delta_j^{(1)} \int_0^t dt' \exp\left[- \int_0^{t'} a_j(\tau)\, d\tau\right] \right\}$$

and second that $\int_0^t a_j(\tau)\, d\tau$ is periodic with period $T_j' \equiv 1/(\Delta_j'^2 - Q^2)^{1/2}$ [recall the argument with regard to $\hat{x}_j(t)$ in (4.7)]. On the basis of these observations, we choose $\Delta_j^{(1)}$ as follows:

$$\Delta_j^{(1)} = \left\langle \left\{ \exp\left[- \int_0^t a_j(\tau)\, d\tau \right] \right\} b_j(t) \right\rangle \Big/ \left\langle \exp\left[- \int_0^t a_j(\tau)\, d\tau \right] \right\rangle \tag{B.11}$$

We have thus determined $\{\Delta_j^{(1)}\}$ for both regimes.

Let us now consider the equation for w_1, which reads

$$\hat{w}_1(t) = -\hat{Z} + N^{-1} \sum_{j=1}^{N} e^{2\pi i' \psi_j(t)}$$
$$+ \lambda \int_0^t dt' \, \{A_-(t, t') \, \hat{w}_1(t') - A_+(t, t') \, \hat{w}_1^*(t')\} - 2\pi i' G(t) \qquad \text{(B.12)}$$

as is derived in the same way as (2.16), where $\hat{w}_1(t) \equiv w_1(t)/\sqrt{N}$, and

$$G(t) \equiv \lim_{N \to \infty} N^{-1} \sum_{j=1}^{N} \hat{\Delta}_j^{(1)} e^{X_j(t)} \int_0^t dt' \exp\left[- \int_0^{t'} a_j(\tau) \, d\tau \right] \qquad \text{(B.13)}$$

with $\hat{\Delta}_j^{(1)} \equiv \Delta_j^{(1)}/\sqrt{N}$ and $X_j(t)$ defined in (4.3). The last term on the rhs of (B.12) makes (B.12) differ from the corresponding equation derived by way of the naive expansion, (2.16). For the subcritical regime we see that $G(t)$ is given by

$$G(t) = \lim_{N \to \infty} N^{-1} \sum_{j=1}^{N} \hat{\Delta}_j^{(1)} e^{2\pi i' (\Delta_j' t + \psi_j(0))} \times t$$
$$\equiv t g(t) \qquad \text{(B.14)}$$

which should vanish as $t \to \infty$ because $g(t)$ is expected to be a rapidly decreasing function for $t \to \infty$ just lilke $A(t)$ in (3.2). Next, for the supercritical regime, we obtain, using the approximation (4.9) for $X_j(t)$ with $t \gg 1$,

$$G(t) \cong t \lim_{N \to \infty} N^{-1} \sum_{|\Delta_j'| > Q} e_j \exp[2\pi i' \operatorname{sgn}(\Delta_j')(\Delta_j'^2 - Q^2)^{1/2} t] \qquad \text{(B.15)}$$

where

$$e_j \equiv \hat{\Delta}_j^{(1)} \left\langle \exp\left[- \int_0^t a_j(\tau) \, d\tau \right] \right\rangle \exp[2\pi i' \psi_j(0)]$$

The rhs of (B.15) also should vanish for $t \to \infty$ for the same reason as mentioned with regard to (B.14). Therefore $G(t)$ is not a dangerous function in both regimes. The problem is that it depends through $\{\hat{\Delta}_j^{(1)}\}$ on $\hat{w}_1$, which is unknown at the moment. In order to solve Eq. (B.12) under this difficulty, let us take a self-consistent approach: we first *assume* that the last term on the rhs of (B.12) is negligible; then we solve (B.12) to obtain $\hat{w}_1$, and finally confirm the validity of the assumption for $G(t)$ with $\{\hat{\Delta}_j^{(1)}\}$ calculated from that $\hat{w}_1$. Let us begin. Since (B.12) is apparently the same as (2.16) when the term with $G(t)$ is left out, we may again find (3.5) for

$\varepsilon < \varepsilon_c$ and (4.21) for $\varepsilon > \varepsilon_c$ as $\hat{w}_1(t)$, where it is necessary to replace Δ_j and $\bar{f}(\Delta)$ in those equations by Δ_j' and its distribution function $\equiv \bar{f}_N(\Delta')$, respectively, because in the present case the ψ_j obey (B.3), not (2.11). It is now easy to evaluate $\{\hat{\Delta}_j^{(1)}\}$. For $\varepsilon < \varepsilon_c$ we obtain by (3.5) and (B.9)

$$\hat{\Delta}_j^{(1)} \cong \frac{\varepsilon}{2\pi N} \operatorname{Im} \left\{ \frac{1}{1 - \lambda \tilde{A}(2\pi \Delta_j)} \right\} \tag{B.16}$$

while for $\varepsilon > \varepsilon_c$ we can show using (4.21) and (B.11) and, invoking the approximation for $X_j(t)$, that

$$\hat{\Delta}_j^{(1)} \approx (1/N)\, W_{\operatorname{sgn}(\Delta_j)}(\Delta_j) e^{-2\pi i' \psi_j(0)} \Big/ \left\langle \exp - \int_0^t a_j(\tau)\, d\tau \right\rangle \tag{B.17}$$

for nonentrained oscillators, where $W_m(\Delta_j)$ stands for the part of (4.21) with the curly brackets $\{\cdots\}$. By (B.16), (B.17), and (B.13) we see that $G(t)$ is $O(N^{-1})$, and hence $-2\pi i' G(t)$ in (B.12) is a negligible term, consistent with the assumption. [It should be passed over to the balance equation of $O(N^{-1})$.] In this way it turns out that the replacements $\Delta_j \to \Delta_j'$, $\bar{f} \to \bar{f}_N$ in (3.5) as well as (4.21) are the only differences caused by the secular terms that have not been taken into account in Section 2.2. It follows from (B.16), (B.17), and (B.13) that $\Delta_j' - \Delta_j$ is always at most $O(N^{-1})$ for every j. Since $\bar{f}_N(\Delta)$ converges to $\bar{f}(\Delta)$ for $N \to \infty$, our results for σ and C_Z, which depend only on $\bar{f}$, remain valid.

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